College of Science
Department of Statistics & OR

OR 122
Introduction to Operations Research

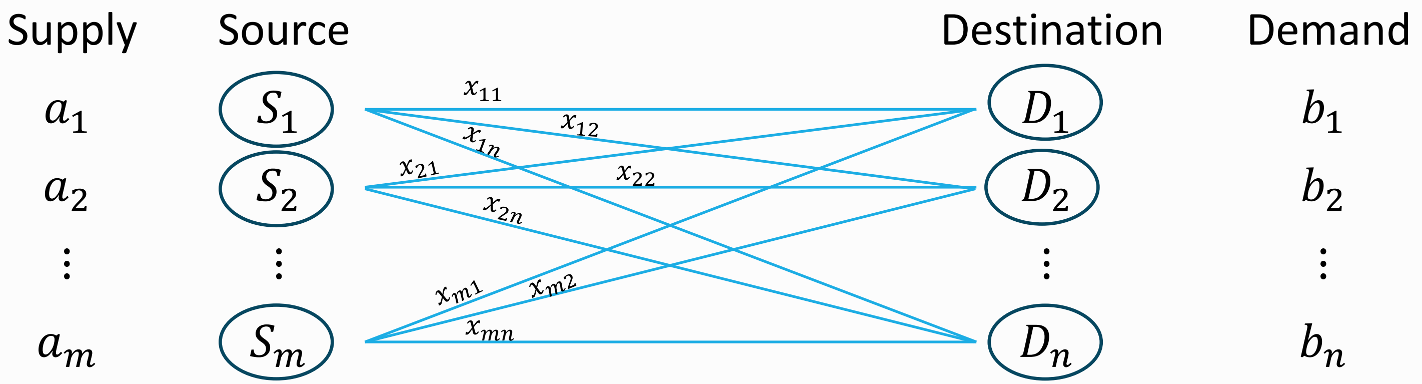
Chapter 5:

Transportation Model

Note: These class notes were originally prepared by Prof. Sameh Asker, Dr. Wael Al Hajailan, Dr. Adel Alrasheedi, and have been subsequently revised and improved by: Dr. Razan Alsehibani, Dr. Kholood Alyazidi and Alanoud Alzughaibi.

Introduction

The transportation problem deals with a special class of linear programming problem in which the objective is to transport a single commodity from several sources to different destinations at a minimum total cost.



Mathematically, the transportation problem may be stated as a LPP as follows:

$$\text{Min Cost} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Capacity Constraint →

Requirement Constraint →

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} x_{ij} \le a_i & ; i = 1,2,\dots,m. \\ \sum_{i=1}^{m} x_{ij} \ge b_j & ; j = 1,2,\dots,n. \\ x_{ij} \ge 0 & ; i = 1,2,\dots,m. \\ & ; j = 1,2,\dots,n. \end{cases}$$

Existence of feasible solution:

A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Which means that:

$$\text{Total capacity} = \text{Total demand (or requiremnt)}$$

Types of Transportation Problem

Transportation Problem

- **Balanced**

  $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

- **Unbalanced**

  $$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

  - If $$\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$$ "Surplus case"
  - If $$\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$$ "Shortage case"

Procedure for solving TP:

1) North-west Corner Rule (NCR).
2) Least-Cost method.
3) Vogel's Approximation.

In this chapter, we will present the first approach.

The algorithmic steps of (NCR):

Step 1: Select the north-west corner cell in the transportation table.

Step 2: Take the min between supply and demand.

Step 3: Adjust the supply and demand values.

The following examples show the application of this method.

Transportation Table

					Supply
	c_{11} x_{11}	c_{12} x_{12}	$\cdots$	c_{1n} x_{1n}	a_1
	c_{21} x_{21}	c_{22} x_{22}	$\cdots$	c_{2n} x_{2n}	a_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	c_{m1} x_{m1}	c_{m2} x_{m2}	$\cdots$	c_{mn} x_{mn}	a_m
Demand	b_1	b_2	$\cdots$	b_n	

c_{ij}: The unit shipping (transportation) cost from source i to destination j.

x_{ij}: The number of transported units from source i to destination j.

$i = 1,2, \ldots, m$ and $j = 1,2, \ldots, n$.

Example

Suppose:

$a_1 = 35, a_2 = 50, a_3 = 40.$

$b_1 = 45, b_2 = 20, b_3 = 30, b_4 = 30.$

$c_{11} = 8, c_{12} = 6, c_{13} = 10, c_{14} = 9.$

$c_{21} = 9, c_{22} = 12, c_{23} = 13, c_{24} = 7.$

$c_{31} = 14, c_{32} = 9, c_{33} = 16, c_{34} = 5.$

This is balanced transportation problem since $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, where $m = 3, n = 4$.

Transportation Table

					Supply			
	8 (35)	6	10	9	35	0		
	9 (10)	12 (20)	13 (20)	7	50	40	20	0
	14	9	16 (10)	5 (30)	40	30	0	
Demand	45	20	30	30				
	10	0	10	0				
	0		0					

Min(35,45)
Min(10,50)
Min(20,40)
Min(20,30)
Min(10,40)
45-35
30-20
50-10
40-20
40-10

So, the basic feasible solution is:

$x_{11} = 35, x_{12} = 0, x_{13} = 0, x_{14} = 0.$

$x_{21} = 10, x_{22} = 20, x_{23} = 20, x_{24} = 0.$

$x_{31} = 0, x_{32} = 0, x_{33} = 10, x_{34} = 30.$

$$z = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34}$$

$$= 8(35) + 6(0) + 10(0) + 9(0) + 9(10) + 12(20) + 13(20) + 7(0) + 14(0) + 9(0) + 16(10) + 5(30)$$

$$= 1180$$

Is this solution optimal?

To answer this question, we need to calculate the following:

- For each occupied cell (i, j) calculate the weight of u_i and v_j so that:

$$u_i + v_j = c_{ij}$$

  - Each row i has a weight u_i and each column j has a weight v_j.
  - Let $u_1 = 0$

- For each unoccupied cell (i, j) calculate:

$$u_i + v_j - c_{ij}$$

  - Let δ_{ij} denote for $u_i + v_j - c_{ij}$ i.e. $\delta_{ij} = u_i + v_j - c_{ij}$.

- If $\delta_{ij} \leq 0$ for all the unoccupied cells, then the solution is optimal.

	$0+v_1=8$ $v_1=8$	$1+v_2=12$ $v_2=11$	$1+v_3=13$ $v_3=12$	$4+v_4=5$ $v_4=1$	Supply
$u_1=0$	[8] 35	[6] $0+11-6$ $\delta_{12}=5$	[10] $0+12-10$ $\delta_{13}=2$	[9] $0+1-9$ $\delta_{14}=-8$	35
$u_2+8=9$ $u_2=1$	[9] 10	[12] 20	[13] 20	[7] $1+1-7$ $\delta_{24}=-5$	50
$u_3+12=16$ $u_3=4$	[14] $4+8-14$ $\delta_{31}=-2$	[9] $4+11-9$ $\delta_{32}=6$	[16] 10	[5] 30	40
Demand	45	20	30	30	

There is $\delta_{ij} > 0$, So the solution is not optimal $(z = 1180)$.

We need to find a correct closed path to optimize the solution.

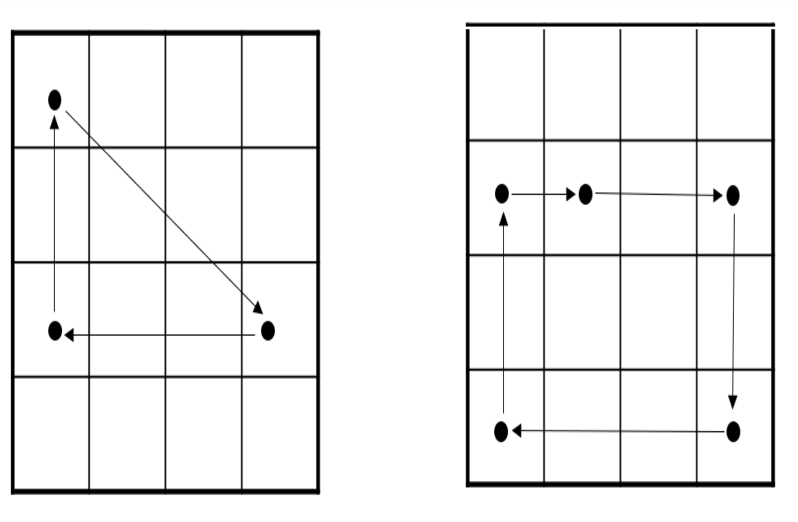
Closed Path

An ordered sequence of at least four different cells is called a loop if:

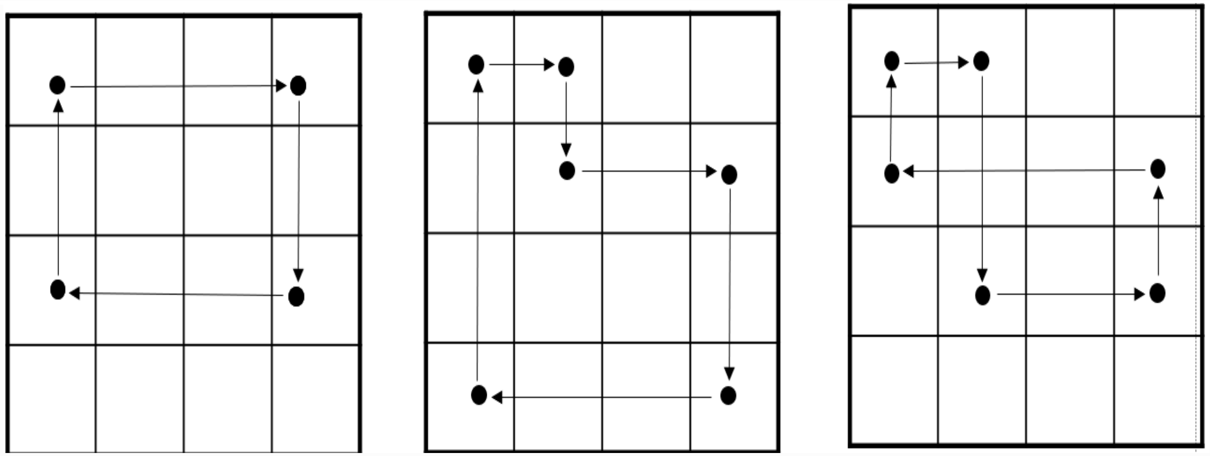
1) Any two consecutive cells lie in either the same row or same column.

2) No three consecutive cells lie in the same row or column.

3) The last cell in the sequence has a row or column in common with the first cell in the sequence.



Incorrect closed path



Correct closed path

The way to find basic feasible solution

1) Determine the largest positive value of δ_{ij}. Let it be δ^*.

$$\delta^* = \max\{\delta_{ij}: \text{unoccipied cell } (i,j)\}$$

2) Create a transformation loop in the transformation table so that:
   - Verify that the conditions of the transformation loop satisfies.
   - The loop contains only one unoccupied cell, which has the value δ^*.

3) Distribute the sign (+) and (-) for all the cells alternatively, starting with the unoccupied cell that has the value δ^*.

4) Select from the cells that have (-) the cell that has the lowest value θ.

$$\theta = \min\{x_{ij}: \text{occupied cell } (i,j) \text{ that has } (-)\}$$

5) Move to the new basic feasible solution, so that the new values for the cells in the transformation loop be as follow:

$$x_{ij} = x_{ij} + \theta \qquad \text{for cells that have } (+)$$
$$x_{ij} = x_{ij} - \theta \qquad \text{for cells that have } (-)$$

The remaining cells stay the same (the values do not change).

Example 1

A company has three factories located at three different cities $F_1, F_2,$ and F_3 which supply three different warehouses $W_1, W_2,$ and W_3. Weekly factory capacities are $200, 160$ and 90 unit, respectively. Weekly warehouses requirements are $180, 120$ and 150 unit, respectively. Unit shipping costs are given in the table below.

Determine the optimum distribution for this company to minimize shipping cost.

	W_1	W_2	W_3
F_1	16	20	12
F_2	14	8	18
F_3	26	24	16

Solution

Construct the TP table as follows:

				Supply
	16	20	12	200
	14	8	18	160
	26	24	16	90
Demand	180	120	150	

Total cost $= 16x_{11} + 20x_{12} + 12x_{13} + 14x_{21} + 8x_{22} + 18x_{23} + 26x_{31} + 14x_{32} + 16x_{33}$

$$\sum_{i=1}^{3} a_i = 200 + 160 + 90 = 450$$

$$\sum_{j=1}^{3} b_j = 180 + 120 + 150 = 450$$

So, we have $\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j \Rightarrow$ This TP is balanced.

Then the basic feasible solution is:

16 / **180**	20 / **20**	12	200 20 0
14	8 / **100**	18 / **60**	160 60 0
26	24	16 / **90**	90 0
180 0	120 100 0	150 90 0	

$$O = \begin{cases} x_{11} = 180, x_{12} = 20, x_{13} = 0 \\ x_{21} = 0, x_{22} = 100, x_{23} = 60 \\ x_{31} = 0, x_{32} = 0, x_{33} = 90 \end{cases}$$

$$\text{Total cost}(O) = 16(180) + 20(20) + 12(0) + 14(0) + 8(100) + 18(60) + 26(0) + 24(0) + 16(90) = 6600$$

Note: Then number of occupied cells at any stage of feasible solutions must equal to (no. of rows + no. of column -1) i.e. $(m + n - 1)$.

For the example above ⇒ No. of occupied cells $= 5$.

No. of rows $= 3$.

No. of columns $= 3$.

Is this solution (O) optimal?

Test for optimality of O:

For the occupied cells:

$u_1 + v_1 = 16, \quad u_1 = 0 \Rightarrow v_1 = 16$

$u_1 + v_2 = 20, \quad u_1 = 0 \Rightarrow v_2 = 20$

$u_2 + v_2 = 8, \quad \Rightarrow u_2 = -12$

$u_2 + v_3 = 18, \Rightarrow \qquad v_3 = 30$

$u_3 + v_3 = 16, \Rightarrow u_3 = -14$

	v_1	v_2	v_3
u_1	16 / 180	20 / 20	12
u_2	14	8 / 100	18 / 60
u_3	26	24	16 / 90

For the unoccupied cells:

If all $\delta \leq 0$ then the solution is optimal.

$$\delta_{13} = u_1 + v_3 - c_{13} = 0 + 30 - 12 = 18$$
$$\delta_{21} = u_2 + v_1 - c_{21} = -12 + 16 - 14 = -10$$
$$\delta_{31} = u_3 + v_1 - c_{31} = -14 + 16 - 26 = -24$$
$$\delta_{32} = u_3 + v_2 - c_{32} = -14 + 20 - 24 = -18$$

It is clear that not all $\delta_{ij} \leq 0,$ so O is not optimum.

Improving the basic solution

Constructing the closed path. Start with the most positive cell. In this case the cell $\delta_{13} = 18$.

16 180	20 20 $(-)$	12 $(+)$
14	8 100 $(+)$	18 60 $(-)$
26	24	16 90

$\Longrightarrow$

	v_1	v_2	v_3
u_1	16 180	20	12 20
u_2	14	8 120	18 40
u_3	26	24	16 90

The new basic solution is :

$$O^* = \begin{cases} x_{11} = 180, x_{12} = 0, x_{13} = 20 \\ x_{21} = 0, x_{22} = 120, x_{23} = 40 \\ x_{31} = 0, x_{32} = 0, x_{33} = 90 \end{cases}$$

Total cost $(O^*) = 16(180) + 20(0) + 12(20) + 14(0) + 8(120) + 18(40) + 26(0) + 24(0) + 16(90) = 6240$

Test for optimality of O^*:

$u_1 + v_1 = 16, \quad u_1 = 0 \Rightarrow v_1 = 16$

$u_1 + v_3 = 12, \quad u_1 = 0 \Rightarrow v_3 = 12$

$u_2 + v_2 = 8,$

$u_2 + v_3 = 18, \Rightarrow u_2 = 6 \Rightarrow v_2 = 2$

$u_3 + v_3 = 16, \Rightarrow u_3 = 4$

	v_1	v_2	v_3
u_1	16 180	20	12 20
u_2	14	8 120	18 40
u_3	26	24	16 90

For the unoccupied cells:

If all $\delta \leq 0$ then the solution is optimal.

$$\delta_{12} = u_1 + v_2 - c_{12} = 0 + 2 - 20 = -18$$
$$\delta_{21} = u_2 + v_1 - c_{21} = 6 + 16 - 14 = 8$$
$$\delta_{31} = u_3 + v_1 - c_{31} = 4 + 16 - 26 = -6$$
$$\delta_{32} = u_3 + v_2 - c_{32} = 4 + 2 - 24 = -18$$

It is clear that not all $\delta_{ij} \leq 0,$ so O^* is not optimum.

Improving the basic solution

Constructing the closed path. Start with the most positive cell. In this case the cell $\delta_{21} = 8$.

16 180 (−)	20	12 20 (+)
14 (+)	8 120	18 40 (−)
26	24	16 90

$\Longrightarrow$

	v_1	v_2	v_3
u_1	16 140	20	12 60
u_2	14 40	8 120	18
u_3	26	24	16 90

So the new improved solution is :

$$O^{**} = \begin{cases} x_{11} = 140, x_{12} = 0, x_{13} = 60 \\ x_{21} = 40, x_{22} = 120, x_{23} = 0 \\ x_{31} = 0, x_{32} = 0, x_{33} = 90 \end{cases}$$

$\text{Total cost } (O^{**}) = 16(140) + 20(0) + 12(60) + 14(40) + 8(120) + 18(0) + 26(0) + 24(0) + 16(90) = 5920$

Test for optimality of O^{**}:

$u_1 + v_1 = 16, \quad u_1 = 0 \Rightarrow v_1 = 16$

$u_1 + v_3 = 12, \quad u_1 = 0 \Rightarrow v_3 = 12$

$u_2 + v_1 = 14, \quad u_2 = -2$

$u_2 + v_2 = 8, \Rightarrow v_2 = 10$

$u_3 + v_3 = 16, \Rightarrow u_3 = 4$

	v_1	v_2	v_3
u_1	cost 16, 180	cost 20	cost 12, 60
u_2	cost 14, 40	cost 8, 120	cost 18
u_3	cost 26	cost 24	cost 16, 90

For the unoccupied cells:

If all $\delta \leq 0$ then the solution is optimal.

$$\delta_{12} = u_1 + v_2 - c_{12} = 0 + 10 - 20 = -10$$

$$\delta_{23} = u_2 + v_3 - c_{23} = -2 + 12 - 18 = -8$$

$$\delta_{31} = u_3 + v_1 - c_{31} = 4 + 16 - 26 = -6$$

$$\delta_{32} = u_3 + v_2 - c_{32} = 4 + 10 - 24 = -10$$

It is obvious that $\delta_{ij} \leq 0,$ then O^{**} is an optimum solution.

Hence, $\text{Total cost } (O^{**}) = 5920$

Example 2

Solve the following transportation problem to maximize profit for the following table.

Destination (Profit per unit)

Origin	1	2	3	4	Supply
A	42	27	24	35	190
B	46	37	32	32	60
C	40	40	30	32	140
Demand	80	30	120	60	

Solution

This is a maximization problem. So, first we must convert this into minimization problem. The conversion of maximization into minimization is done by subtracting the unit profit of the table from the highest unit profit. From the table 46 is the highest unit profit.

First Construct the loss matrix and write the objective function.

$$\text{Max} \quad z = 42x_{11} + 27x_{12} + 24x_{13} + 35x_{14} + 46x_{21} + 37x_{22} + 32x_{23} + 32x_{24} + 40x_{31} + 40x_{32} + 30x_{33} + 32x_{34}$$

The Loss Matrix

The largest profit $= 46$

4	19	22	11
0	9	14	14
6	6	16	14

The total supply $= 390$, the total demand $= 290$. So it is unbalanced transportation problem, surplus $= 390 - 290 = 100$.

	v_1	v_2	v_3	v_4	v_5	Supply
u_1	4 80	19 30	22 80	11	0	190 110 80 0
u_2	0	9	14 40	14 20	0	60 20 0
u_3	6	6	16	14 40	0 100	140 100 0
	80	30	120	60	100	
	0	0	40	40	0	
			0	0		

The initial solution is:

$$O = \begin{cases} x_{11} = 80, x_{12} = 30, x_{13} = 80, x_{14} = 0, x_{15} = 0 \\ x_{21} = 0, x_{22} = 0, x_{23} = 40, x_{24} = 20, x_{25} = 0 \\ x_{31} = 0, x_{32} = 0, x_{33} = 0, x_{34} = 40, x_{35} = 100 \end{cases}$$

$\text{The no. of occupied cells} = 7$

$\text{No. of rows} + \text{no. of columns} - 1 = 3 + 5 - 1 = 7$

Test for optimality of O:

$$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$$

$$u_1 + v_2 = 19, \quad \Rightarrow v_2 = 19$$

$$u_1 + v_3 = 22, \quad \Rightarrow v_3 = 22$$

$$u_2 + v_3 = 14, \Rightarrow u_2 = -8$$

$$u_2 + v_4 = 14, \quad \Rightarrow v_4 = 22$$

$$u_3 + v_4 = 14, \Rightarrow u_3 = -8$$

$$u_3 + v_5 = 0 \quad \Rightarrow v_5 = 8$$

$$\delta_{14} = u_1 + v_4 - c_{14} = 0 + 22 - 11 = 11$$

$$\delta_{15} = 0 + 8 - 0 = 8$$

$$\delta_{21} = -8 + 4 - 0 = -4$$

$$\delta_{22} = -8 + 19 - 9 = 2$$

$$\delta_{25} = -8 + 8 - 0 = 0$$

$$\delta_{31} = -8 + 4 - 6 = -10$$

$$\delta_{32} = -8 + 19 - 6 = 5$$

$$\delta_{33} = -8 + 22 - 16 = -2$$

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19 30	22 80(-)	11 (+)	0
u_2	0	9	14 40(+)	14 20(-)	0
u_3	6	6	16	14 40	0 100

Since, not all $\delta_{ij} \leq 0$, then O is not an optimum solution.

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19 30	22 60	11 20	0
u_2	0	9	14 60	14	0
u_3	6	6	16	14 40	0 100

The new basic solution is :

$$O^* = \begin{cases} x_{11} = 80, x_{12} = 30, x_{13} = 60, x_{14} = 20, x_{15} = 0 \\ x_{21} = 0, x_{22} = 0, x_{23} = 60, x_{24} = 0, x_{25} = 0 \\ x_{31} = 0, x_{32} = 0, x_{33} = 0, x_{34} = 40, x_{35} = 100 \end{cases}$$

Test for optimality of O^*:

$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$

$u_1 + v_2 = 19, \quad \Rightarrow v_2 = 19$

$u_1 + v_3 = 22, \quad \Rightarrow v_3 = 22$

$u_1 + v_4 = 11, \quad \Rightarrow v_4 = 11$

$u_2 + v_3 = 14, \Rightarrow u_2 = 14 - 22 = -8$

$u_3 + v_4 = 14, \Rightarrow u_3 = 3$

$u_3 + v_5 = 0 \quad \Rightarrow v_5 = -3$

$$\delta_{15} = u_1 + v_5 - c_{15} = 0 - 3 - 0 = -3$$

$$\delta_{21} = -8 + 4 - 0 = -4$$

$$\delta_{22} = -8 + 19 - 9 = 2$$

$$\delta_{24} = -8 + 11 - 14 = -11$$

$$\delta_{25} = -8 - 3 - 0 = -11$$

$$\delta_{31} = 3 + 4 - 6 = 1$$

$$\delta_{32} = 3 + 19 - 6 = 16$$

$$\delta_{33} = 3 + 22 - 16 = 9$$

	v_1	v_2	v_3	v_4	v_5
u_1	4 / 80	19 / 30(-)	22 / 60	11 / 20(+)	0
u_2	0	9	14 / 60	14	0
u_3	6	6 / (+)	16	14 / 40(-)	0 / 100

Since, not all $\delta_{ij} \leq 0$, then O^* is not an optimum solution.

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19	22 60	11 50	0
u_2	0	9	14 60	14	0
u_3	6	6 30	16	14 10	0 100

Test for optimality of O^{**}:

$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$

$u_1 + v_3 = 22, \quad \Rightarrow v_3 = 22$

$u_1 + v_4 = 11, \quad \Rightarrow v_4 = 11$

$u_2 + v_3 = 14, \Rightarrow u_2 = 14 - 22 = -8$

$u_3 + v_4 = 14, \Rightarrow u_3 = 3$

$u_3 + v_5 = 0, \Rightarrow v_5 = -3$

$\delta_{12} = 0$

$\delta_{15} = -3$

$\delta_{21} = -4$

$\delta_{22} = -14$

$\delta_{24} = -11$

$\delta_{25} = -11$

$\delta_{31} = 1$

$\delta_{33} = 9$

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19	22 60(-)	11 50(+)	0
u_2	0	9	14 60	14	0
u_3	6	6 30	16 (+)	14 10(-)	0 100

Since, not all $\delta_{ij} \leq 0$, then $O^{**}$is not an optimum solution.

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19	22 50	11 60	0
u_2	0	9	14 60	14	0
u_3	6	6 30	16 10	14	0 100

Test for optimality of O^{***}:

$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$

$u_1 + v_3 = 22, \quad \Rightarrow v_3 = 22$

$u_1 + v_4 = 11, \quad \Rightarrow v_4 = 11$

$u_2 + v_3 = 14, \Rightarrow u_2 = -8$

$u_3 + v_3 = 16, \quad \Rightarrow u_3 = -6$

$u_3 + v_2 = 6, \Rightarrow v_2 = 12$

$u_3 + v_5 = 0, \Rightarrow v_5 = 6$

$\delta_{12} = -7$

$\delta_{15} = 6$

$\delta_{21} = -4$

$\delta_{22} = -5$

$\delta_{24} = -11$

$\delta_{25} = -2$

$\delta_{31} = -8$

$\delta_{34} = -9$

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19	22 50(-)	11 60	0 (+)
u_2	0	9	14 60	14	0
u_3	6	6 30	16 10(+)	14	0 100(-)

Since, not all $\delta_{ij} \le 0$, then O^{***} is not an optimum solution.

	v_1	v_2	v_3	v_4	v_5
u_1	4 80	19	22	11 60	0 50
u_2	0	9	14 60	14	0
u_3	6	6 30	16 60	14	0 50

Test for optimality of O^{****}:

$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$

$u_1 + v_4 = 11, \quad \Rightarrow v_4 = 11$

$u_1 + v_5 = 0, \quad \Rightarrow v_5 = 0$

$u_3 + v_5 = 0, \quad \Rightarrow u_3 = 0$

$u_3 + v_2 = 6, \quad \Rightarrow v_2 = 6$

$u_3 + v_3 = 16, \Rightarrow v_3 = 16$

$u_2 + v_3 = 14, \Rightarrow u_2 = -2$

$\delta_{12} = -13$

$\delta_{13} = -6$

$\delta_{21} = 2$

$\delta_{22} = -5$

$\delta_{24} = -5$

$\delta_{25} = -2$

$\delta_{31} = -2$

$\delta_{34} = -3$

	v_1	v_2	v_3	v_4	v_5
u_1	4 80(-)	19	22	11 60	0 50(+)
u_2	0 (+)	9	14 60(-)	14	0
u_3	6	6 30	16 60(+)	14	0 50(-)

Since, not all $\delta_{ij} \leq 0$, then O^{****} is not an optimum solution.

	v_1	v_2	v_3	v_4	v_5
u_1	4 30	19	22	11 60	0 100
u_2	0 50	9	14 10	14	0
u_3	6	6 30	16 110	14	0

Test for optimality of O^{*****}:

$u_1 + v_1 = 4, \quad u_1 = 0 \Rightarrow v_1 = 4$

$u_1 + v_4 = 11, \quad \Rightarrow v_4 = 11$

$u_1 + v_5 = 0, \quad \Rightarrow v_5 = 0$

$u_2 + v_1 = 0, \Rightarrow u_2 = -4$

$u_2 + v_3 = 14, \Rightarrow v_3 = 18$

$u_3 + v_3 = 16, \Rightarrow u_3 = -2$

$u_3 + v_2 = 6, \quad \Rightarrow v_2 = 8$

$\delta_{12} = -11$

$\delta_{13} = -4$

$\delta_{22} = -5$

$\delta_{24} = -7$

$\delta_{25} = -4$

$\delta_{31} = -4$

$\delta_{34} = -5$

$\delta_{35} = -2$

Hence, all $\delta_{ij} \leq 0$, then O^{*****} is an optimum solution.

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The optimum solution is :

$$O^{*****} = \begin{cases} x_{11} = 30, x_{12} = 0, x_{13} = 0, x_{14} = 60, x_{15} = 100 \\ x_{21} = 50, x_{22} = 0, x_{23} = 10, x_{24} = 0, x_{25} = 0 \\ x_{31} = 0, x_{32} = 30, x_{33} = 110, x_{34} = 0, x_{35} = 0 \end{cases}$$

$$\text{Max} \quad \text{proift} = 42(30) + 35(60) + 46(50) + 32(10) + 40(30) + 30(110)$$

$$= 10480$$