College of Science
Department of Statistics & OR

OR 122
Introduction to Operations Research

Chapter 4:

The simplex Algorithm

Note: These class notes were originally prepared by Prof. Sameh Asker, Dr. Wael Al Hajailan, Dr. Adel Alrasheedi, and have been subsequently revised and improved by: Dr. Razan Alsehibani, Dr. Kholood Alyazidi and Alanoud Alzughaibi.

Before we get through into the algorithm of simplex we show how to convert on LPP to the standard form.

To convert LPP into standard form, each inequality constraint must be replaced by an equality constraint. Here, for the “less than or equal” constraint we add a new variable that is called a slack variable.

Slack variable:

Is denoted by s_i for the constraint i^{th} and represents the unused amount of the resource in this constraint.

For example, the constraint $x_1 + x_2 \leq 40$ is converted into

$x_1 + x_2 + s_1 = 40, \qquad s_1 \geq 0$

Excess variable:

Sometimes it is called surplus variable and is denoted by e_i for the "greater than or equal" constraints.

For example, the constraint $x_1 + x_2 \geq 40$ is converted into

$$x_1 + x_2 - e_1 = 40, \qquad e_1 \geq 0$$

Basic and Non-basic variables:

Consider a system $Ax = b$ of m linear equations and n variables. Assuming $n \geq m$ then a basic solution to $Ax = b$ is obtained by setting $n - m$ variables equal to zero and solving for the values of the remaining m variables.

In this case, we set the non-basic variables (or NBV) equal to zero. Therefore, the other remaining variables are called basic variables.

Example 1

Find the basic solution for the system

$$\left.\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}\right\} \Rightarrow m = 2 \text{ and } n = 3$$

So we have $n - m = 3 - 2 = 1$ non-basic variables equal to zero.

Let $x_3 = 0 \Rightarrow x_2 = 1$ and $x_1 = 2$.

Then the basic solution $= (2,1,0)$

Note: some set of m variables do not give a basic solution.

Example 2

$$\left.\begin{matrix} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + x_3 = 3 \end{matrix}\right\} \Rightarrow m = 2 \text{ and } n = 3$$

Then the non-basic variables $= 1$.

Let $x_3 = 0 \Rightarrow \left.\begin{matrix} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{matrix}\right\} \Rightarrow \triangle = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0,\quad \triangle_{x_1} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0, x_1 = \frac{-2}{0}$

Then the system has no basic solution if we set $x_3 = 0$.

Now, let $x_2 = 0 \Rightarrow \left.\begin{matrix} x_1 + x_3 = 1 \\ 2x_1 + x_3 = 3 \end{matrix}\right\} \Rightarrow \triangle = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

$$\triangle_{x_1} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 1 - 3 = -2,\quad \triangle_{x_3} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$x_1 = \frac{-2}{-1} = 2, x_3 = \frac{1}{-1} = -1 \Rightarrow$The basic solution $= (2,0,-1)$.

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Feasible Basic solution:

A basic solution is called feasible basic solution if all its variable are non-negative (≥ 0)

For example, the solution $(2,1,0)$ is feasible basic solution while the solution $(2,0,-1)$ fails to be feasible.

Note: For a LPP in the standard form with n variable and m equality constraints a set of $n-m$ non-basic variables (or equivalently, m basic variables) can be chosen as follows

$$\binom{n}{m} = \frac{n!}{m!\ (n-m)!}$$

Example 3

Find the basic solutions for the following

$$x_1 + 2x_2 \leq 6$$
$$2x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Solution: Convert to the standard form as follows

$$\left.\begin{array}{r} x_1 + 2x_2 + s_1 = 6 \quad (*) \\ 2x_1 + x_2 + s_2 = 8 \quad (**) \\ x_1, x_2, s_1, s_2 \geq 0 \end{array}\right\} \Rightarrow m = 2 \text{ and } n = 4$$

Then the non-basic variables $= n - m = 2.$

The chosen set of non-basic variables $\binom{n}{m} = \binom{4}{2} = \frac{4!}{2!\ (4-2)!} = 6.$

How can we evaluate?

Remain variables | Remain variables | Remain variables

$s_1 = 0$ → (1) $s_2 = 0$, (2) $x_1 = 0$, (3) $x_2 = 0$

$s_2 = 0$ → (4) $x_1 = 0$, (5) $x_2 = 0$

$x_1 = 0$ → (6) $x_2 = 0$

Then the set is $\Rightarrow$

Case (1) $\Rightarrow$ The non-basic variables are $s_1 = 0, s_2 = 0 \Rightarrow x_1 = 3.33, \ x_2 = 1.33$.

$\Rightarrow$ The solution $= (3.33, 1.33, 0, 0)$. Feasible basic solution.

Case (2) $\Rightarrow s_1 = 0, x_1 = 0 \Rightarrow x_2 = 3, \quad s_2 = 5.$

$\Rightarrow (0,3,0,5)$. Feasible basic solution.

Case (3) $\Rightarrow s_1 = 0, x_2 = 0 \Rightarrow x_1 = 6, \quad s_2 = -4.$

$\Rightarrow (6,0,0,-4)$. Non-feasible $s_2 < 0$.

Case (4) $\Rightarrow s_2 = 0, x_1 = 0 \Rightarrow x_2 = 8, \quad s_1 = -10.$

$\Rightarrow (0,8,-10,0)$. Non-feasible $s_1 < 0$.

Case (5) $\Rightarrow s_2 = 0, x_2 = 0 \Rightarrow x_1 = 4, \quad s_1 = 2.$

$\Rightarrow (4,0,2,0)$. Feasible basic solution.

Case (6) $\Rightarrow x_1 = 0, x_2 = 0 \Rightarrow s_1 = 6, \quad s_2 = 8.$

$\Rightarrow (0,0,6,8)$. Feasible basic solution.

Therefore, the above results can be summarized as follows:

Case	Non-basic variables	Basic variables	Solution	FBS	NF
(1)	s_1, s_2	x_1, x_2	$(3.33, 1.33,0,0)$	√	
(2)	s_1, x_1	s_2, x_2	$(0, 3,0,5)$	√	
(3)	s_1, x_2	s_2, x_1	$(6, 0,0, -4)$		√
(4)	s_2, x_1	s_1, x_2	$(0, 8, -10,0)$		√
(5)	s_2, x_2	s_1, x_1	$(4, 0,2,0)$	√	
(6)	x_1, x_2	s_1, s_2	$(0, 0,6,8)$	√	

Simplex algorithm for Max problem:

Step 1: Convert the LPP to standard form.

Step 2: Find a basic feasible solution. For the constraints " $\leq$ " with non-negative right-hand sides, we can use the slack variables s_i as the basic variables.

Step 3: If the non-basic variables in row 0 have non-negative coefficients, then the current basic feasible solution is optimal . If any variable have negative coefficient in row 0, choose the variable with the most negative coefficient in row 0 to enter the basic. We call this variable the entering variable.

Step 4: use elementary row operations to make the entering variable the basic variable in any row that wins the ratio test. After the elementary row operations have been used to create a new canonical form, return to step 3.

We use the following example to apply the simplex algorithm.

Example 4

$$\text{Max} \quad Z = 10x_1 + 6x_2 + 4x_3$$

$$\text{Subject to} \quad \begin{cases} x_1 + x_2 + x_3 \leq 100 \\ 10x_1 + 4x_2 + 5x_3 \leq 600 \\ 2x_1 + 2x_2 + 6x_3 \leq 300 \end{cases}$$

$$x_1, x_2, x_3 \geq 0$$

Solution

The standard form,

$$
\begin{aligned}
x_1 + x_2 + x_3 + s_1 \qquad\qquad &= 100 \\
10x_1 + 4x_2 + 5x_3 \qquad + s_2 \qquad &= 600 \\
2x_1 + 2x_2 + 6x_3 \qquad\qquad + s_3 &= 300
\end{aligned}
$$

$$\text{and} \quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Then the standard form become,

$$\text{Max} \quad Z = 10x_1 + 6x_2 + 4x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

$$\text{Subject to} \quad \begin{cases} x_1 + x_2 + x_3 + s_1 = 100 \\ 10x_1 + 4x_2 + 5x_3 + s_2 = 600 \\ 2x_1 + 2x_2 + 6x_3 + s_3 = 300 \end{cases}$$

$$x_i, \; s_i \geq 0 \, , i = 1,2,3.$$

To build the initial table, we should write the objective function, and the constraints as follows:

Max Z

s.t.

$$Z - 10x_1 - 6x_2 - 4x_3 = 0$$
$$x_1 + x_2 + x_3 + s_1 = 100$$
$$10x_1 + 4x_2 + 5x_3 + s_2 = 600$$
$$2x_1 + 2x_2 + 6x_3 + s_3 = 300$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The initial table

Basic variable ↓ (BV) Right hand side ↓ (RHS)

	BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS
R_0: Objective function row →	Z	-10	-6	-4	0	0	0	0
R_1: 1st constraint row →	s_1	1	1	1	1	0	0	100
R_2: 2nd constraint row →	s_2	10	4	5	0	1	0	600
R_3: 3rd constraint row →	s_3	2	2	6	0	0	1	300

The current feasible basic solution is: $(0,0,0,100,600,300)$ and $z = 0$.

But this solution is not optimal since row 0 contains negative variables.

So, one of the NBV will enter to be BV and one of the BV will leave.

- The entry NBV is the most negative (when objective is max).
- The entry NBV is the most positive (when objective is min).
- The Leaving BV is the one that has a smaller ratio test.
- The intersection of the entry variable and the leaving is called the pivotal element.

Ratio test = RHS/coefficient of entry variable

Entry variable

BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS	RT
Z	-10	-6	-4	0	0	0	0	
s_1	1	1	1	1	0	0	100	100
s_2	10	4	5	0	1	0	600	60
s_3	2	2	6	0	0	1	300	150

Leaving variable → s_2

Minimum RT ← 60

Pivotal element: 10

- The pivotal element must be equal to 1.
- Each cell above and under the pivotal element must be equal to 0.

	BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS
	Z							
	$\boldsymbol{s_1}$							
$\div 10$ →	$\boldsymbol{x_1}$	1	0.4	0.5	0	0.1	0	60
	$\boldsymbol{s_3}$							

The new $R_2 = \text{old } R_2 \div 10$

BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	0	-2	1	0	1	0	600
s_1	0	0.6	0.5	1	-0.1	0	40
x_1	1	0.4	0.5	0	0.1	0	60
s_3	0	1.2	5	0	-0.2	1	180

The new $R_0 = (10 \times \text{new } R_2) + old\ R_0$
The new $R_1 = (-1 \times \text{new } R_2) + old\ R_1$
The new $R_3 = (-2 \times \text{new } R_2) + old\ R_3$

The current feasible basic solution is: $(60, 0, 0, 40, 0, 180)$ and $z = 600$.
But this solution is not optimal since row 0 contains negative variables.

entry variable

leaving variable → s_1 row; Minimum RT ← 40/0.6 = 66.67; Pivotal element → 0.6

BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS	RT
Z	0	-2	1	0	1	0	600	
s_1	0	0.6	0.5	1	-0.1	0	40	$40/0.6= 66.67$
x_1	1	0.4	0.5	0	0.1	0	60	$60/0.4 = 150$
s_3	0	1.2	5	0	-0.2	1	180	$180/1.2 = 150$

Pivotal element

The new $R_1 = oldR_1 \div 0.6$

BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	0	0	16/6	20/6	4/6	0	2200/3
x_2	0	1	5/6	10/6	-1/6	0	400/6
x_1	1	0	1/6	-2/3	1/6	0	100/3
s_3	0	0	4	-2	0	1	100

The new $R_0 = (2 \times \text{new } R_1) + old\ R_0$
The new $R_2 = (-0.4 \times \text{new } R_1) + old\ R_2$
The new $R_3 = (-1.2 \times \text{new } R_1) + old\ R_3$

The current feasible basic solution is: $(100/3, 400/6, 0, 0, 0, 100)$ and $z = 2200/3 \approx 733.33$.
This solution is optimal since row 0 does not contain negative variables.

- In row 0 in ALL simplex table, the coefficient of BV is equal to 0.

- In row 0 in FINAL simplex table, for the optimal solution:

➢ If the values of all NBV are greater than zero, then we have a single (unique) optimal solution.

➢ If there is a NB variable equal to zero, then we have multiple optimal basic solutions. When wanting to find an alternative optimal solution, one of the NB variables whose value is equal to zero is chosen to become BV.

- When it is not possible to perform the minimum ratio test for all rows, i.e. there is no positive value (greater than zero) in the column of the entry variable. We conclude that the optimal solution is infinite (unbounded), that is, $z^* = +\infty$ when the objective function is max, and $z^* = -\infty$ when the objective function is min.

Example 5

Use the simplex method to solve the following LPP:

$$\text{Min} \quad Z = 2x_1 - 3x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 + x_2 \leq 4 \\ x_1 - x_2 \leq 6 \end{cases}$$

$$x_1, x_2 \geq 0$$

Solution

The standard form,

$$x_1 + x_2 + s_1 = 4$$
$$x_1 - x_2 + s_2 = 6$$

$$\text{Min} \quad Z = 2x_1 - 3x_2$$

Subject to $\begin{cases} x_1 + x_2 + s_1 = 4 \\ x_1 - x_2 + s_2 = 6 \end{cases}$

$$x_i,\ s_i \geq 0\ ,\ i = 1,2.$$

The initial basic feasible solution is $x_1 = 0, x_2 = 0, s_1 = 4, s_2 = 6$.

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	-2	3	0	0	0	
s_1	0	1	1	1	0	4	4/1= 4
s_2	0	1	-1	0	1	6	-

BV	z	x_1	x_2	s_1	s_2	RHS
Z	1	-5	0	-3	0	-12
x_2	0	1	1	1	0	4
s_2	0	2	0	1	1	10

It is clear in the last table that all the coefficients for NBV in row 0 are non-positive then the optimum solution is $x_1 = 0, x_2 = 4, s_1 = 0, s_2 = 10$, and . Min $z = -12$.

Note: The above example "Min" can be solved using the rotation Min $z = -\text{Max}(-z)$.

Example 6

Solve using the simplex approach. (This example has multiple optimum solution).

$$\text{Max} \quad Z = -3x_1 + 6x_2$$

$$\text{Subject to} \quad \begin{cases} 5x_1 + 7x_2 \leq 35 \\ -x_1 + 2x_2 \leq 2 \end{cases}$$

$$x_1, x_2 \geq 0$$

Solution

The standard form,

$$\text{Max} \quad Z = -3x_1 + 6x_2$$

$$\text{Subject to} \quad \begin{cases} 5x_1 + 7x_2 + s_1 = 35 \\ -x_1 + 2x_2 + s_2 = 2 \end{cases}$$

$$x_i,\ s_i \geq 0\ , i = 1,2.$$

The initial basic feasible solution is $x_1 = 0, x_2 = 0, s_1 = 35, s_2 = 2$.

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	3	-6	0	0	0	
s_1	0	5	7	1	0	35	35/7= 5
s_2	0	-1	2	0	1	2	2/2= 1

BV	z	x_1	x_2	s_1	s_2	RHS
Z	1	0	0	0	3	6
s_1	0	17/2	0	1	-7/2	28
x_2	0	-1/2	1	0	1/2	1

From the last table we have all the coefficients of row 0 are non-negative then $\text{Max } z = 6$ and $(0,1,28,0)$ is the optimum solution.

Note: Since $x_1$is NBV and its coefficient in row 0 is equal to 0, so we have multiple solutions. Another optimum solution with $\text{Max } z = 6$ can be obtained by assuming x_1 as an entry variable and find RT of BV to find the leaving one and complete as before.

So we have $(\frac{56}{17}, \frac{45}{17}, 0,0)$ gives $\text{Max } z = 6$.

Note: Another optimum solution that gives Max $z = 6$ can be obtained by averaging the two solutions that we got $(x_1 = 0, x_2 = 1)$ and $(x_1 = \frac{56}{17}, x_2 = \frac{45}{17})$.

The average solution $\Rightarrow$

$$x_1 = \frac{\left(0 + \frac{56}{17}\right)}{2} = \frac{28}{17}$$

$$x_2 = \frac{\left(1 + \frac{45}{17}\right)}{2} = \frac{31}{17}$$

The other optimum solution is $\left(\frac{28}{17}, \frac{31}{17}\right)$ gives Max $z = 6$.

Example 7

Show that the following example has unbonded solution using the simplex approach:

$$\text{Max} \quad Z = 2x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 - x_2 \leq 4 \\ -x_1 + x_2 \leq 1 \end{cases}$$

$$x_1, x_2 \geq 0$$

The standard form,

$$\text{Max} \quad Z = 2x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 - x_2 + s_1 = 4 \\ -x_1 + x_2 + s_2 = 1 \end{cases}$$

$$x_i, \ s_i \geq 0 \,, i = 1,2.$$

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The initial feasible solution is $s_1 = 4, s_2 = 1$.

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	0	-2	0	0	0	
s_1	0	1	-1	1	0	4	-
s_2	0	-1	1	0	1	1	1

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	-2	0	0	2	2	
s_1	0	0	0	1	1	5	-
x_2	0	-1	1	0	1	1	-

From the last table we have coefficient below -2 is less than or equal to zero then we can't compute the RT then we have unbounded solution.

Example 8

Solve the following LPP using the simplex approach:

$$Min \quad Z = -2x_1 - 3x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 - x_2 \leq 1 \\ x_1 - 2x_2 \geq 2 \end{cases}$$

$$x_1, x_2 \geq 0$$

The initial basic feasible solution is $s_1 = 1, s_2 = 2$.

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	2	3	0	0	0	
s_1	0	1	-1	1	0	1	-
s_3	0	1	-2	0	1	2	-

From the above table, we have an unbounded solution.

Degenerate LLP:

A LPP is called degenerate if it has at least one basic feasible solution in which a basic variable is equal to zero.

Example 9

Show that the following LPP is degenerate:

$$\text{Max} \quad Z = 5x_1 + 2x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 + x_2 \leq 6 \\ x_1 - x_2 \leq 0 \end{cases}$$

$$x_1, x_2 \geq 0$$

The initial basic feasible solution is $x_1 = 0, x_2 = 0(\text{non} - \text{basic variable})$.
$s_1 = 6, s_2 = 0$ (basic variable) $\Longrightarrow$(Degenerate LPP since s_2 is BV $= 0$)

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	-5	-2	0	0	0	
s_1	0	1	1	1	0	6	6/1=6
s_2	0	1	-1	0	1	0	0/1=0

← minimum

BV	z	x_1	x_2	s_1	s_2	RHS	RT
Z	1	0	-7	0	5	0	
s_1	0	0	2	1	-1	6	6/2=3
x_1	0	1	-1	0	1	0	-

negative

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BV	z	x_1	x_2	s_1	s_2	RHS
Z	1	0	0	3.5	1.5	21
x_2	0	0	1	1/2	-1/2	3
x_1	0	1	0	1/2	1/2	3

From the last table, we get all the coefficients in row 0 are positive then the optimum solution is $x_1 = 3, x_2 = 3, s_1 = 0, s_2 = 0$ and Max $z = 21$.

From the previous example we get,

- From the initial table we have the wining ratio $= 0$. This means that after x_1 enters the basic, x_1 will be zero in the new basic feasible solution (in table1).

- The new feasible solution with $x_1 = 0$ will have $z = 0$ as the old one (table 0).