College of Science
Department of Statistics & OR

OR 122
Introduction to Operations Research

# Chapter 2:

## Sensitivity Analysis

**Note:** These class notes were originally prepared by Prof. Sameh Asker, Dr. Wael Al Hajailan, Dr. Adel Alrasheedi, and have been subsequently revised and improved by: Dr. Razan Alsehibani, Dr. Kholood Alyazidi and Alanoud Alzughaibi.

Definition:

Sensitivity analysis is concerned with how changes in LPP's parameters affect the optimum solution.

- Types of LPP coefficients:

1) Coefficients of the objective function:

$Z = c_1x_1 + c_2x_2$

The coefficients here are $c_1$ and $c_2$.

2) Coefficients of the constraints :

$a_{11}x_1 + a_{12}x_2 \leq b_1$

$a_{21}x_1 + a_{22}x_2 \geq b_2$

The coefficients here are $b_1$ and $b_2$.

Before we study the influence of these parameters on the LPP's optimal solutions let us define the following:

Bending Constraint:

The constraint $a_{11}x_1 + a_{12}x_2 \leq b_1$ or $a_{11}x_1 + a_{12}x_2 \geq b_1$ is said to be a bending constraint if $a_{11}x_1^* + a_{12}x_2^* = b_1$,
where $(x_1^*, x_2^*)$ is the optimum solution.

Non-bending Constraint:

The constraint $a_{11}x_1 + a_{12}x_2 \leq b_1$ or $a_{11}x_1 + a_{12}x_2 \geq b_1$ is said to be a non-bending constraint if $a_{11}x_1^* + a_{12}x_2^* < b_1$ or $a_{11}x_1^* + a_{12}x_2^* > b_1$.

Note:

A bending constraint is always associated with a rare resource, while the non-bending constraint is associated with available resources.

The following example shows how to understand sensitivity analysis.

# Example 1

A firm produce two types of paints, exterior and interior paints. To produce a unit from each of the two paints, two basic materials must be mixed. These two raw materials are A and B. the firm can support 6 tons from material A and 8 tons from material B each week at most. The following table shows the tons required to produce 1 unit from each types.

| | E | I | Available |
|---|---|---|---|
| A | 1 | 2 | 6 |
| B | 2 | 1 | 8 |

The market prerequisites show that the demand on the interior type cannot exceed the demand on the exterior type with at most 1 ton. Furthermore, the studies on market show that the demand on the interior type cannot exceed 2 tons. The firm wants to get the optimum solution that satisfies the maximum profit for the firm if it sells the ton of interior type with 2000 SR and for the exterior is 3000 SR.

# Solution

$x_1$: the number producing tons of exterior paint.

$x_2$: the number producing tons of interior paint.

The mathematical form of the LPP is:

$$\text{Max} \quad Z = 3000x_1 + 2000x_2$$

$$\text{Subject to} \quad \begin{cases} x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 8 \\ x_2 \leq 1 + x_1 \Leftrightarrow x_2 - x_1 \leq 1 \\ x_2 \leq 2 \end{cases}$$

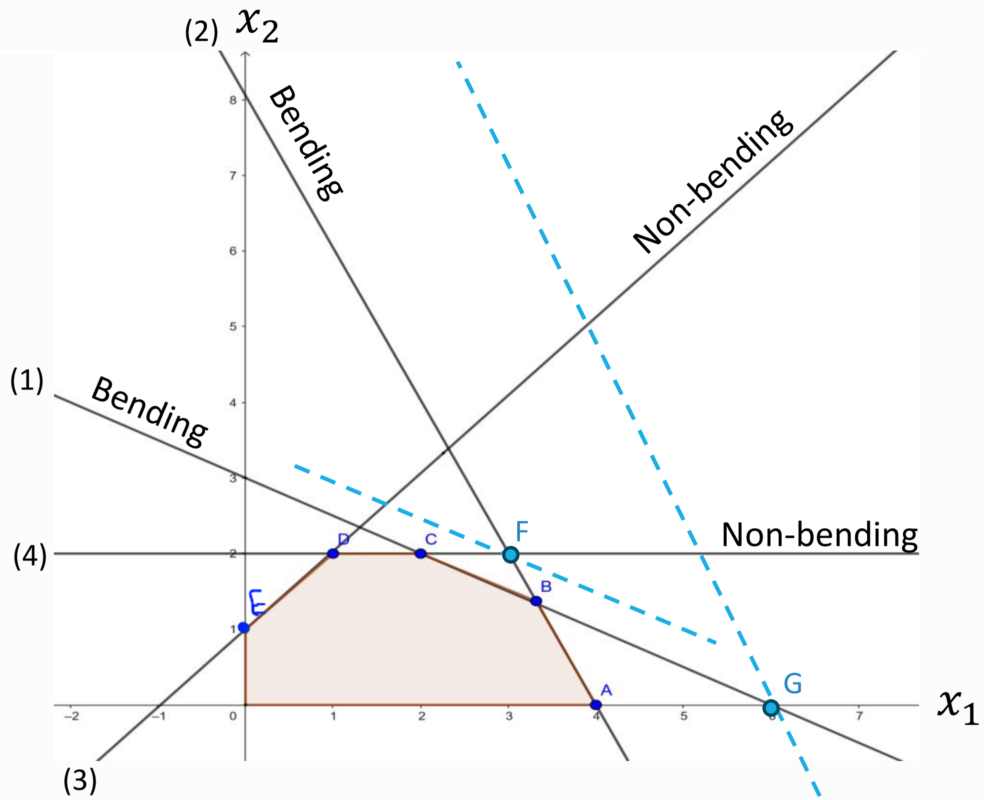
$$x_1, \ x_2 \geq 0$$

The feasible solution region = O A B C D E

The optimum solution = B

$B = \left(\frac{10}{3}, \frac{4}{3}\right)$

$Z|_B = 12666.7$



Discuss the sensitivity analysis for the above example.

First: we classify the constraints.

Constraint (1) $\Rightarrow$ L.H.S $\left(\frac{10}{3},\frac{4}{3}\right)=\frac{10}{3}+2\left(\frac{4}{3}\right)=\frac{18}{3}=6=$ R.H.S

$\Rightarrow$ (1) is a bending constraint (rare resource)

Constraint (2) $\Rightarrow$ L.H.S $\left(\frac{10}{3},\frac{4}{3}\right)=2\left(\frac{10}{3}\right)+\frac{4}{3}=\frac{24}{3}=8=$ R.H.S

$\Rightarrow$ (2) is a bending constraint (rare resource)

Constraint (3) $\Rightarrow$ L.H.S $\left(\frac{10}{3},\frac{4}{3}\right)=\frac{4}{3}-\frac{10}{3}=-\frac{6}{3}=-2<$ R.H.S

$\Rightarrow$ (3) is a non-bending constraint (available resource)

Constraint (4) $\Rightarrow$ L.H.S $\left(\frac{10}{3},\frac{4}{3}\right)=\frac{4}{3}=\frac{4}{3}<$ R.H.S

$\Rightarrow$ (4) is a non-bending constraint (available resource)

Second: To what extent should we increase the rare resources associated with the first constraint to improve the value of the objective function?

We should move the constraint (1) parallel to itself until it pass through the point $F$ which will be the new optimum solution.

Calculating $F \Rightarrow$ The new optimum solution is $F = (3,2)$.

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$Z(F) = 3000\ (3) + 2000\ (2) = 13000\ SR$.

The new form of (1) ⇒ L.H.S $(3, 2) = 3 + 2(2) = 7$

The new constraint is $x_1 + 2x_2 \leq 7$

$\triangle_1 =$ The maximum increasing $= 7 - 6 = 1$ ton.

The shadow price of constraint (1) $= \frac{Z_{new} - Z_{old}}{\triangle_1} = \frac{13000 - 12666.7}{1} = 333.3$ SR.

The sensitivity analysis for the bending constraint (2):

The constraint (2) should move parallel to itself until it passes through the point $(6, 0)$.

The new optimum solution is G$= (6,0)$.

The new maximum of objective $\Rightarrow Z(G) = 3000\ (6) + 2000\ (0) = 18000\ SR$.

The new form of (2) $\Rightarrow$ L.H.S $(6,0) = 2(6) + 0 = 12$

The new constraint is $2x_1 + x_2 \leq 12$

$\triangle_2 =$ The maximum increasing $= 12 - 8 = 4$ tons.

The shadow price of constraint (2) $= \frac{18000-12666.7}{4} = 1333.325$ SR

The sensitivity analysis for the constraint (3) [available resource]:
The constraint (3) should move parallel to itself until it passes through the old optimum solution $B$.

The new form of (3) $\Rightarrow$ L.H.S$\left(\frac{10}{3},\frac{4}{3}\right) = -2$
The new form is $x_2 - x_1 \leq -2$

Then the minimum reduced value $= 1 - (-2) = 3$ tons.
The shadow price of constraint (3) $= 0$

Note:
Always the shadow price of non-bending constraint is equal to zero since $Z_{new} = Z_{old}$.

The sensitivity analysis for the constraint (4) [available resource]:

The constraint (4) should move parallel to itself until it passes through the old optimum solution $B$.

The new form of (3) $\Rightarrow$ L.H.S$\left(\frac{10}{3}, \frac{4}{3}\right) = \frac{4}{3}$

The new form is $x_2 \leq \frac{4}{3}$

The minimum reduced value $= 2 - \frac{4}{3} = \frac{2}{3}$ tons.

The sensitivity analysis for the coefficients of the objective function.

The typical question is this case is: What is the range of values of the objective coefficients for which the optimum basis remains.

First: The slope of $Z = c_1x_1 + c_2x_2$ $\Rightarrow x_2 = \left(\frac{Z}{c_2}\right) - \left(\frac{c_1x_1}{c_2}\right) \longrightarrow$ (*)

is slope $= -\frac{c_1}{c_2}$

The slope of the bending constraint (1) is $-\frac{1}{2}$

The slope of the bending constraint (2) is $-\frac{2}{1}$

Only take the slope for bending constraint

To study the sensitivity of the objective function we should have

$-2 \leq -\frac{c_1}{c_2} \leq -\frac{1}{2}$ or $\frac{1}{2} \leq \frac{c_1}{c_2} \leq 2$

To find the slope write the equation in (*) form the slope is $x_1$ coefficient.

Then the range of $c_1$ when $c_2 = 2000$ can be calculated as follows:

$$\frac{1}{2} \le \frac{c_1}{2000} \le 2 \Rightarrow 1000 \le c_1 \le 4000$$

This means that at $c_1 \in [1000, 4000]$ the optimum solution will be $\mathrm{B} = \left(\frac{10}{3}, \frac{4}{3}\right)$.

The range of $c_2$ when $c_1 = 3000 \Rightarrow \quad \frac{1}{2} \le \frac{3000}{c_2} \le 2 \quad \Rightarrow \frac{1}{2} \le \frac{c_2}{3000} \le 2 \Rightarrow 1500 \le c_2 \le 6000$

This means that at $c_2 \in [1500, 6000]$ the optimum solution will be $\mathrm{B} = \left(\frac{10}{3}, \frac{4}{3}\right)$.

# Example 2

Test the sensitivity analysis for the following problem.

$$\text{Min} \quad Z = 4\,x_1 + 3\,x_2$$

$$\text{Subject to} \quad \begin{cases} 2\,x_1 + x_2 \geq 10 \\ -3x_1 + 2x_2 \leq 6 \\ x_1 + x_2 \geq 6 \end{cases}$$

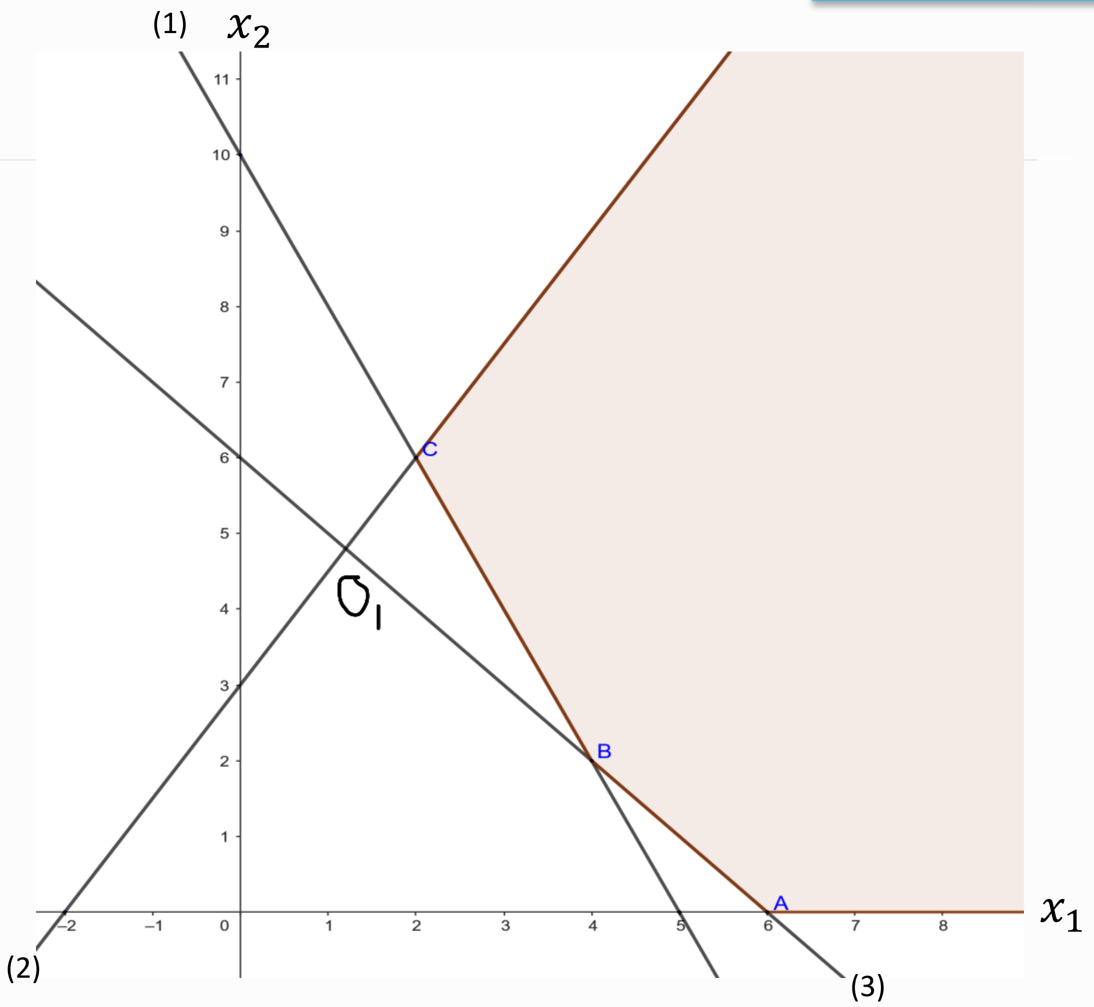
$$x_1,\ x_2 \geq 0$$

# Solution

$2\,x_1 + x_2 \geq 10 \qquad (1)$

$-3x_1 + 2x_2 \leq 6 \qquad (2)$

$x_1 + x_2 \geq 6 \qquad (3)$




In equation (1) we have $(0, 10)$ & $(5, 0)$.

In equation (2) we have $(0, 3)$ &$(-2\ 0)$.

In equation (3) we have $(0, 6)$ & $(6, 0)$.

The points A, B, and C are:

$\text{A} = (6,0)$, $\text{B} = (4,2)$, $\text{C} = (2,6)$.

$Z(A) = 4(6) + 3(0) = 24.$

$Z(B) = 4(4) + 3(2) = 22.$

$Z(C) = 4(2) + 3(6) = 26.$

Min $Z = 22$ , the optimum solution is B.

Sensitivity Analysis:

Constraint (1) $\Rightarrow$ L.H.S $(4,2) = 2(4) + 2 = 10 =$ R.H.S (Bending)

Constraint (2) $\Rightarrow$ L.H.S $(4,2) = -3(4) + (2)2 = -12 + 4 = -8 <$ R.H.S (Non-bending)

Constraint (3) $\Rightarrow$ L.H.S $(4,2) = 4 + 2 = 6 =$ R.H.S (Bending)

Regarding Constraint (1):

It should move to the point $O_1, O_1 = \left(\frac{6}{5}, \frac{24}{5}\right)$.

The new form of (1) $\Rightarrow$ L.H.S $(O_1) = 2\left(\frac{6}{5}\right) + \frac{24}{5} = \frac{12+24}{5} = \frac{36}{5} = 7.2$

The new constraint is $2x_1 + x_2 \geq 7.2$

The minimum increasing $= 10 - 7.2 = 2.8$

$Z(O_1) = 4\left(\frac{6}{5}\right) + 3\left(\frac{24}{5}\right) = \frac{24}{5} + \frac{72}{5} = \frac{96}{5} = 19.2.$

Shadow price $= \frac{22-19.2}{2.8} = 1$ SR.

Regarding Constraint (2):

It should be moved to the old point $\mathrm{B} = (4, 2)$.

The new constraint is $-3x_1 + 2x_2 \leq -8$

The maximum decreasing $= 6 - (-8) = 14$ tons

Regarding Constraint (3):
It should move to pass through the point $(5,0)$

$Z(5,0) = 4(5) + 3(0) = 20.$

The new constraint is $x_1 + x_2 \geq 5$

The minimum increasing $= 6 - 5 = 1$ ton

Shadow price $= \frac{22-20}{1} = 2$ SR.

The shadow price must be non-negative since it is a price.

The sensitivity analysis for the coefficients of the objective function.

The slope of good $Z = -\frac{c_1}{c_2}$

The slope of the bending constraint (1) is $-\frac{2}{1} = -2$

The slope of the bending constraint (3) is $-\frac{1}{1} = -1$

Then we have $-2 \leq -\frac{c_1}{c_2} \leq -1$

If $c_1 = 4 \Rightarrow \quad 1 \leq \frac{4}{c_2} \leq 2 \quad \Rightarrow \frac{1}{2} \leq \frac{c_2}{4} \leq 1 \Rightarrow c_2 \in [2, 4]$

If $c_2 = 3 \Rightarrow \quad 1 \leq \frac{c_1}{3} \leq 2 \quad \Rightarrow c_1 \in [3, 6]$

# Example 3

A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods A and B are available at a cost SR 4 and SR 3 per unit, respectively. If one unit of A contains 200 unit of vitamins, 1 unit of mineral and 40 calories and 1 unit of B contains 100 units of vitamins, 2 units of minerals and 40 calories.

- Find by the graphical method what combination of foods be used to have least cost?
- Study the sensitivity analysis for this diet problem.

# Solution

Let us first summarize the given data in the following table:

| Decision Variables | Food | Unit content of | | | Cost per unit |
|---|---|---|---|---|---|
| | | Vitamin | Minerals | Calories | |
| $x_1$ | A | 200 | 1 | 40 | 4 |
| $x_2$ | B | 100 | 2 | 40 | 3 |
| Minimum required | | 4000 | 50 | 1400 | |

$x_1$: The number of units of food A.

$x_2$: The number of units of food B.

Now the mathematical form can be given as follows,

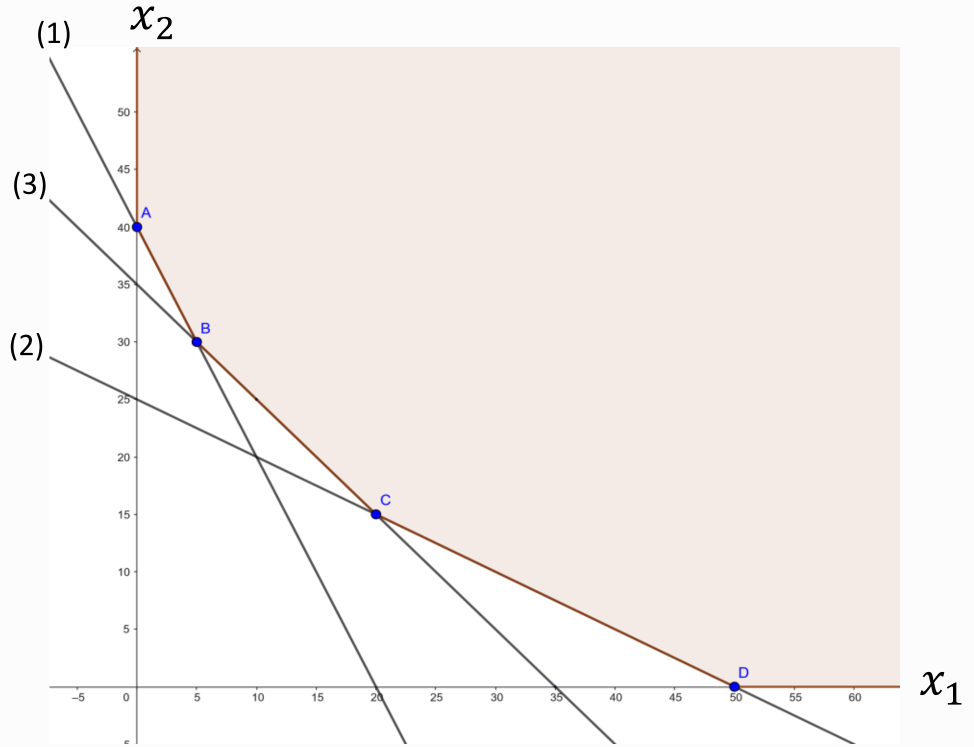
$$\text{Minimum the cost} \quad Z = 4\,x_1 + 3\,x_2$$

$$\text{Subject to} \quad \begin{cases} 200\,x_1 + 100\,x_2 \geq 4000 \\ x_1 + 2x_2 \geq 50 \\ 40\,x_1 + 40\,x_2 \geq 1400 \end{cases} \Longrightarrow \quad \text{Simplified} \begin{cases} 2\,x_1 + x_2 \geq 40 \\ x_1 + 2x_2 \geq 50 \\ x_1 + x_2 \geq 35 \end{cases}$$

$$x_1,\ x_2 \geq 0 \qquad\qquad x_1,\ x_2 \geq 0$$

The points A, B, C, and D are:

$A = (0, 40)$, $B = (5, 30)$, $C = (20, 13)$, $D = (50, 0)$.




The optimum solution is B.

$Z(B) = 110.$

And the diet should contain 5 units of food A and 30 units of food B.

Now we are going to study the sensitivity analysis. First, we classify the constraints as follows:

Constraint (1) $\Rightarrow$ L.H.S $(5,30) = 2(5) + 3 = 40 =$ R.H.S $\Rightarrow$ Bending constraint (rare resource)

Constraint (2) $\Rightarrow$ L.H.S $(5,30) = 5 + 60 = 65 >$ R.H.S $\Rightarrow$ Non-bending constraint (available resource)

Constraint (3) $\Rightarrow$ L.H.S $(5,30) = 5 + 30 = 35 =$ R.H.S $\Rightarrow$ Bending constraint (rare resource)

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The sensitivity analysis for the bending constraint (1) can be obtained by moving this constraint parallel to itself until it passes through the point $(0, 35)$. The new optimum solution will be this point and the new form for the constraint is given by:

The new form of (1) ⇒ L.H.S. $(0, 35) = 2(0) + 35 = 35$

The new constraint is $2x_1 + x_2 \geq 35$

The minimum increased of the vitamins $= 40 - 35 = 5$ units.

$\text{Cost } (0, 35) = 4(0) + 3(35) = 105.$

Shadow price $= \frac{110-105}{5} = \frac{5}{5} = 1$ SR.

Regarding the bending constraint (3)it should be moved parallel to itself until it passes through the point $(10, 20)$. So the new form for the constraint is then given by:

The new form of (1) ⇒ L.H.S. $(10, 20) = 10 + 20 = 30$

The new constraint is $x_1 + x_2 \geq 30$

The minimum increased of the calories $= 35 - 30 = 5$ units.

Cost $(10, 20) = 4(10) + 3(20) = 100.$

Shadow price $= \frac{110-100}{5} = \frac{10}{5} = 2$ SR.

For the non-bending constraint, it should be moved to pass through the old optimum solution B and then it’s new form becomes,

The new constraint is $x_1 + 2x_2 \geq 65$

The minimum increased of the minerals $= 65 - 50 = 15$ units.

The sensitivity analysis for the coefficients of the cost can be given as follows:

The slope of the cost $Z = c_1x_1 + c_2x_2$ is $-\frac{c_1}{c_2}$

The slope of the bending constraint (1) $= -2$

The slope of the bending constraint (3) $= -1$

Then we have $-2 \leq -\frac{c_1}{c_2} \leq -1$

Which can be simplified as,

$1 \leq \frac{c_1}{c_2} \leq 2$

Suppose we fix $c_1$ to $4$ then $1 \leq \frac{4}{c_2} \leq 2 \quad \Rightarrow \frac{1}{2} \leq \frac{c_2}{4} \leq 1 \quad \Rightarrow 2 \leq c_2 \leq 4 \quad \Rightarrow c_2 \in [2, 4]$

Suppose we fix $c_2$ to $3$ then $1 \leq \frac{c_1}{3} \leq 2 \quad \Rightarrow 3 \leq c_1 \leq 6 \quad \Rightarrow c_1 \in [3, 6]$

This means that when $c_2 \in [2, 4]$ and fixing $c_1 = 4$ the optimum solution B will not be changed and only the objective function will be improved. The same observation is for the parameter $c_2$.