| OPER 441 | Final Exam | First term 1444 |
|---|---|---|

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| الاسم : | الرقم: |
|---|---|

**Problem 1 :** Let $T$ be the random variable that give the period of time (in months) between two rainfalls in Abha city and its CDF given by. ليكن $T$ المتغير العشوائي الذي يقيس الزمن بين هطولين للمطر بالأشهر في مدينة أبها.

$$F(t) = 2.12\left(\frac{6}{10} - e^{\frac{-t}{2}}\right)$$

a) Find the inverse transform function $t = F^{-1}(u)$.

$$u = F(t) \Rightarrow t = F^{-1}(u) = -2\ln\left(\frac{1.272 - u}{2.12}\right)$$

b) Use the random numbers in the tableau to determine the time of the 8th rainfall.

حدد زمن هطول المطر للمرة الثامنة مستعينا بتكملة الجدول التالي .

| $k$ | $u_k$ | $t_k = F^{-1}(u_k)$ | $x_i = \sum_{k=1}^{i} t_k$ |
|---|---|---|---|
| 1 | 0.967 | 3.8777 | 3.8777 |
| 2 | 0.453 | 1.9022 | 5.7799 |
| 3 | 0.381 | 1.7337 | 7.5135 |
| 4 | 0.752 | 2.8107 | 10.3242 |

| $k$ | $u_k$ | $t_k = F^{-1}(u_k)$ | $x_i = \sum_{k=1}^{i} t_k$ |
|---|---|---|---|
| 5 | 0.835 | 3.1585 | 13.4827 |
| 6 | 0.663 | 2.4947 | 15.9774 |
| 7 | 0.916 | 3.5685 | 19.5459 |
| 8 | 0.484 | 1.9793 | 21.5252 |

**Problem 2 :** Consider the following probability density function :

$$f(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 1 \\ \frac{3}{4} - \frac{x}{4}, & 1 \le x \le 3 \end{cases}$$

Determine the CDF $F(x)$ of this distribution.

$$F(x) = \begin{cases} 0 & , \quad x \leqq 0 \\ \frac{1}{2}x & , \quad 0 \leqq x \leqq 1 \\ 1 - \frac{(3-x)^2}{8} & , \quad 1 \leqq x \leqq 3 \\ 1 & , \quad x \geqq 3 \end{cases}$$

**Problem 3 :** Consider the following cumulative density function (CDF):

$$G(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}x, & 0 \le x \le 1 \\ 1 - \frac{(3-x)^2}{8}, & 1 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

1. Determine the inverse transform $x = G^{-1}(u)$ of $G(x)$

$$G^{-1}(u) = \begin{cases} 2u & , \quad 0 \leqq u \leqq 0.5 \\ 3 - \sqrt{8(1-u)} & , \quad 0 \leqq u \leqq 1 \end{cases}$$

2. Generate the values of $X$ using the following sequence of (0,1) random numbers.

| $u$ | 0.387 | 0.336 | 0.466 | 0.074 | 0.184 | 0.34 |
|---|---|---|---|---|---|---|
| $x = G^{-1}(u)$ | 0.774 | 0.672 | 0.932 | 0.144 | 0.368 | 0.68 |
| $u$ | 0.9 | 0.875 | 0.475 | 0.64 | | |
| $x = G^{-1}(u)$ | 2.10 | 2 | 0.95 | 1.302 | | |

**Problem 4 :** Consider the discrete random variable $X$ with the following distribution

$$P(X = x) = \frac{7-k}{21}, \quad k = 1,2,3,4,5,6$$

1. Fill the table (إملأ الجدول )

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X=k)$ | 0.285 | 0.238 | 0.190 | 0.142 | 0.095 | 0.047 |
| $F(x)$ | 0.285 | 0.523 | 0.714 | 0.856 | 0.952 | 1.00 |

2. Find the inverse transform function $F^{-1}(u)$ to generate observations for the random variable $X$

$$F^{-1}(u) = \begin{cases} 1 & , \ 0 \leq u \leq 0.285 \\ 2 & , \ 0.285 < u \leq 0.523 \\ 3 & , \ 0.523 < u \leq 0.714 \\ 4 & , \ 0.714 < u \leq 0.856 \\ 5 & , \ 0.856 < u \leq 0.952 \\ 6 & , \ 0.952 < u \leq 1.00 \end{cases}$$

**Problem 5 :** Consider the following function

$$f(x) = \frac{x^3}{20}, \quad 1 \leq x \leq 3,$$

1. Find the greatest value $f_{max}$ of the function $f(x)$ حدد أكبر قيمة للدالة

$f'(x) = \frac{3}{20}x^2$ always $> 0$ in the interval $[1,3]$

Therefore $f_{max} = f(3) = \frac{3}{20} \times 9 = \frac{27}{20} = 1.35$

2. Use Acceptance/Rejection method to generate 10 observations $(x_i, y_i)$ using the following formulas

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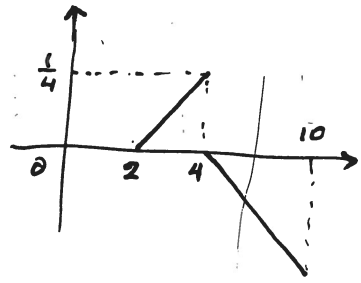
$$x_i = a + u_i(b-a) \ , \ y_i = r_i f_{max}$$

| $i$ | $u_i$ | $x_i$ | $f(x_i)$ | $r_i$ | $y_i = r_i f_{max}$ | $acc./rej.$ |
|---|---|---|---|---|---|---|
| 1 | 0.622 | 2.244 | 0.564987 | 0.311 | 0.420 | ACCEPT |
| 2 | 0.943 | 2.886 | 1.201874 | 0.964 | 1.301 | REJECT |
| 3 | 0.851 | 2.702 | 0.986339 | 0.827 | 1.116 | REJECT |
| 4 | 0.592 | 2.184 | 0.520868 | 0.186 | 0.251 | ACCEPT |
| 5 | 0.084 | 1.168 | 0.079671 | 0.165 | 0.223 | REJECT |
| 6 | 0.936 | 2.872 | 1.184468 | 0.684 | 0.923 | ACCEPT |
| 7 | 0.016 | 1.032 | 0.054955 | 0.768 | 1.037 | REJECT |
| 8 | 0.219 | 1.438 | 0.148678 | 0.667 | 0.900 | REJECT |
| 9 | 0.091 | 1.182 | 0.08257 | 0.257 | 0.347 | REJECT |
| 10 | 0.238 | 1.476 | 0.160779 | 0.280 | 0.378 | REJECT |

**Problem 6 :** Consider the following function

$$f(x) = \begin{cases} \frac{x}{8} - \frac{1}{4}, & 2 \le x \le 4 \\ \frac{3}{4} - \frac{x}{4}, & 4 \le x \le 10 \end{cases}$$




1. Find the greatest value $f_{max}$ of the function $f(x)$ حدد أكبر قيمة للدالة

$$f_{max} = \frac{1}{4} = 0.25$$

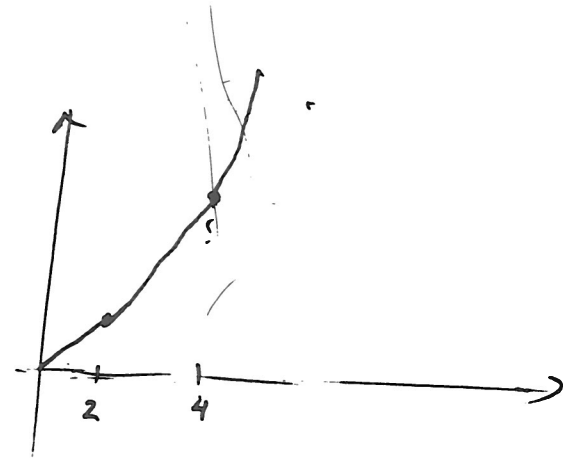
2. Use Acceptance/Rejection method to generate 5 observations $(x_i, y_i)$ using the following formulas

استعمل طريقة " قبول / رفض" لتوليد 10 مشاهدات بتطبيق القوانين التالية

$$x_i = a + u_i(b - a) \ , \ y_i = r_i f_{max}$$

| $i$ | $u_i$ | $x_i$ | $f(x_i)$ | $r_i$ | $y_i = r_i f_{max}$ | $acc./rej.$ |
|---|---|---|---|---|---|---|
| 1 | 0.622 | 6.976 | -0.994 | 0.311 | 0.078 | REJECT |
| 2 | 0.943 | 9.544 | -1.636 | 0.964 | 0.241 | REJECT |
| 3 | 0.851 | 8.808 | -1.452 | 0.827 | 0.207 | REJECT |
| 4 | 0.592 | 6.736 | -0.934 | 0.186 | 0.047 | REJECT |
| 5 | 0.084 | 2.672 | 0.084 | 0.165 | 0.041 | ACCEPT |

$$\begin{cases} \frac{x}{8} - \frac{1}{4} \\ -\frac{3}{4} + \frac{x}{4} \end{cases}$$




$f_{max} = 1.75$

| | | | | | |
|---|---|---|---|---|---|
| 0.622 | 6.976 | 0.994 | 0.311 | 0.544 | ACCEPT |
| 0.943 | 9.544 | 1.636 | 0.964 | 1.687 | REJECT |
| 0.851 | 8.808 | 1.452 | 0.827 | 1.447 | ACCEPT |
| 0.592 | 6.736 | 0.934 | 0.186 | 0.326 | ACCEPT |
| 0.084 | 2.672 | 0.084 | 0.165 | 0.289 | REJECT |

**Problem 7 :** Consider the following cumulative density function (CDF):

$$G(x) = \begin{cases} 0, & x \le 1 \\ \frac{1}{80}(x^4 - 1), & 1 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

and its inverse transform $x = G^{-1}(u) = \sqrt[4]{80u + 1}$ , $1 \le x \le 3$ .

In a mechanic shop (ميكانيكي), suppose that there is 40 % chance that the repair time for a car is a random variable, $X_1$, with CDF $G(x)$, and 60 % chance that the repair time is a random variable, $X_2$, distributed according to an exponential distribution with mean 2 hours.

Complete the table to generate observations from the mixed distribution.

املأ الجدول لتوليد المشاهدات من التوزيع المختلط.

| $n$ | الأرقام عشوانية $U_1(0,1)$ | تحديد $F_{X_1}(x)$ أو $F_{X_2}(x)$ | الأرقام عشوانية $U_2(0,1)$ | المشاهدات $x$ |
|---|---|---|---|---|
| 1 | 0.902 | $F_{X_2}$ | 0.641 | 2.04 |
| 2 | 0.818 | $F_{X_2}$ | 0.984 | 8.27 |
| 3 | 0.375 | $F_{X_1}$ | 0.495 | 2.52 |
| 4 | 0.341 | $F_{X_1}$ | 0.592 | 2.63 |
| 5 | 0.812 | $F_{X_2}$ | 0.815 | 3.37 |
| 6 | 0.148 | $F_{X_1}$ | 0.111 | 1.77 |
| 7 | 0.509 | $F_{X_2}$ | 0.665 | 2.18 |
| 8 | 0.653 | $F_{X_2}$ | 0.379 | 0.95 |