Course Notes: Math 280

Contents

Chapter 1

Real numbers and their bounded subsets

Order Relations on the Real Numbers

Definition 1 (Greater and Less Relations) *For real numbers $x, y \in \mathbb{R}$ we define:*

$$x > y \;:\Longleftrightarrow\; x - y > 0,$$

$$x < y \;:\Longleftrightarrow\; y > x,$$

$$x \geq y \;:\Longleftrightarrow\; (x > y) \;\; \textit{or} \;\; (x = y),$$

$$x \leq y \;:\Longleftrightarrow\; (x < y) \;\; \textit{or} \;\; (x = y).$$

Let $x, y, z, a, b \in \mathbb{R}$. The following properties hold:

(3.1) **Trichotomy.** For any two real numbers x, y, exactly one of the following is true:

$$x < y, \quad x = y, \quad y < x.$$

This property allows us to define

$$\max(x, y) = \begin{cases} x, & x \geq y, \\ y, & \text{otherwise}, \end{cases} \qquad \min(x, y) = \begin{cases} x, & x \leq y, \\ y, & \text{otherwise}. \end{cases}$$

(3.2) **Transitivity.** If $x < y$ and $y < z$, then $x < z$.

Proof: From $x < y$ we have $y - x > 0$, and from $y < z$ we have $z - y > 0$. Adding gives $(z-y)+(y-x) = z - x > 0$, hence $x < z$. □

(3.3) **Translation Invariance.** If $x < y$, then

$$a + x < a + y.$$

Proof: Since $(a + y) - (a + x) = y - x > 0$, it follows that $a + x < a + y$. □

(3.4) **Reflection.**

$$x < y \quad \Longleftrightarrow \quad -x > -y.$$

Proof: $y - x = (-x) - (-y)$. Thus $y - x > 0$ if and only if $-x > -y$. □

(3.5) **Combination with Another Inequality.** If $x < y$ and $a < b$, then

$$x + a < y + b.$$

Proof: From $x < y$ we get $x + a < y + a$. From $a < b$ we get $y + a < y + b$. By transitivity, $x + a < y + b$. □

(3.6) **Multiplication with a Positive Number.** If $x < y$ and $a > 0$, then

$$ax < ay.$$

Proof: Since $y - x > 0$ and $a > 0$, multiplying gives $a(y - x) = ay - ax > 0$, hence $ax < ay$. □

(3.7) If $0 \leq x < y$ and $0 \leq a < b$, then

$$ax < by.$$

Proof: If $x = 0$ or $a = 0$, then $ax = 0 < by$. Otherwise, $0 < x < y$ and $0 < a < b$. From (3.6) we get $ax < ay$ and also $ay < by$. By transitivity, $ax < by$. □

(3.8) **Multiplication with a Negative Number.** If $x < y$ and $a < 0$, then

$$ax > ay.$$

Proof: Since $-a > 0$, from (3.6) we get $(-a)x < (-a)y$. This means $-ax < -ay$, which is equivalent to $ax > ay$. □

(3.9) **Squares are Positive.** For every $x \neq 0$, we have

$$x^2 > 0, \quad \text{in particular } 1 > 0.$$

Proof: If $x > 0$, then $x^2 > 0$ by (3.6). If $x < 0$, then $-x > 0$ and $(-x)^2 = x^2 > 0$. Since $1 = 1^2 \neq 0$, we get $1 > 0$. □

(3.10) **Positive Numbers and Their Inverses.**

$$x > 0 \quad \Longleftrightarrow \quad x^{-1} > 0.$$

Proof: If $x > 0$, then $x^2 > 0$ (by (3.9)). Multiplying $x > 0$ with $x^{-2} > 0$ gives $x^{-1} > 0$. Conversely, if $x^{-1} > 0$, then applying the same argument to x^{-1} yields $x > 0$. □

(3.11) **Inverses Reverse Inequalities.** If $0 < x < y$, then

$$x^{-1} > y^{-1}.$$

Proof: Since $x < y$ and $x, y > 0$, we have $y/x > 1$. Taking reciprocals gives $x^{-1} > y^{-1}$. □

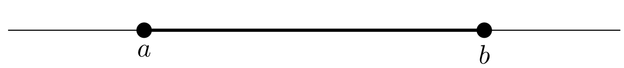
Subsets of $\mathbb{R}$

Interval notation on $\mathbb{R}$

Throughout, let $a, b \in \mathbb{R}$.

1. **Closed intervals.** If $a \leq b$,

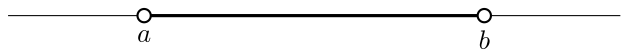
$$[a, b] := \{\, x \in \mathbb{R} : a \leq x \leq b \,\}.$$



For $a = b$, $[a, a] = \{a\}$.

2. **Open intervals.** If $a < b$,

$$(a, b) := \{\, x \in \mathbb{R} : a < x < b \,\}.$$



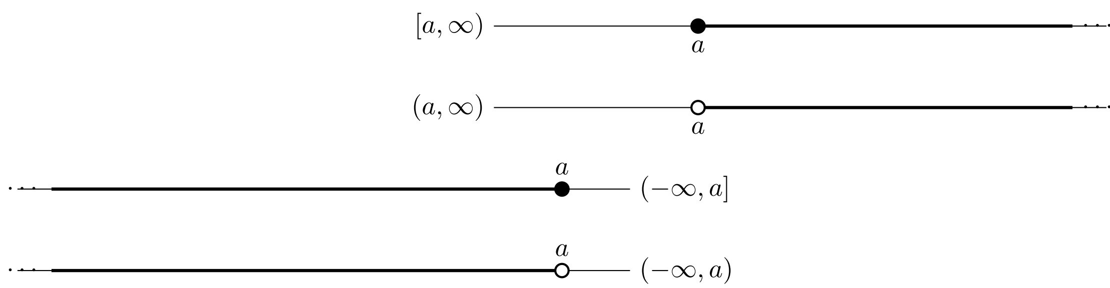
3. **Half-open (half-closed) intervals.** For $a < b$,

$$[a, b) := \{\, x \in \mathbb{R} : a \le x < b \,\}, \qquad (a, b] := \{\, x \in \mathbb{R} : a < x \le b \,\}.$$

4. **Unbounded intervals.** For $a \in \mathbb{R}$,

$$[a, \infty) := \{\, x \in \mathbb{R} : x \ge a \,\}, \qquad (a, \infty) := \{\, x \in \mathbb{R} : x > a \,\},$$

$$(-\infty, a] := \{\, x \in \mathbb{R} : x \le a \,\}, \qquad (-\infty, a) := \{\, x \in \mathbb{R} : x < a \,\}.$$



Example.1

Let $\varepsilon = 0.00001$. Find a natural number $n \in \mathbb{N}$ such that

$$\frac{1}{n+2} < \varepsilon.$$

We require

$$\frac{1}{n+2} < 0.00001.$$

Taking reciprocals (since $n + 2 > 0$), this inequality is equivalent to

$$n + 2 > \frac{1}{0.00001}.$$

Since $\frac{1}{0.00001} = 100000$, we obtain

$$n + 2 > 100000.$$

Thus,

$$n > 99998.$$

Hence, any natural number $n \ge 99999$ satisfies the condition.

Example 2

Let $\varepsilon = 0.00001$. Find $n \in \mathbb{N}$ such that
$$\frac{3n-2}{n^3+2n} < \varepsilon.$$

For $n \geq 1$ we have
$$\frac{3n-2}{n^3+2n} \leq \frac{3n}{n^3} = \frac{3}{n^2}.$$

Thus it suffices to ensure
$$\frac{3}{n^2} < \varepsilon \iff n^2 > \frac{3}{\varepsilon} = 300000 \iff n > \sqrt{300000} \approx 547.72.$$

Hence any $n \geq 548$ works. In particular,
$$\frac{3n-2}{n^3+2n} \leq \frac{3}{n^2} \leq \frac{3}{548^2} < \frac{3}{300000} = 10^{-5} = \varepsilon.$$

Therefore one can take, for example, $\boxed{n = 548}$ (and any larger n also satisfies the inequality).

The Absolute Value

For a real number $x \in \mathbb{R}$, its (absolute) value is defined by
$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Equivalently,
$$|x| = \max(x, -x).$$

Properties of Absolute Value in $\mathbb{R}$ For all $x, y \in \mathbb{R}$, the following hold:

1. $|x| \geq 0$, and $|x| = 0 \iff x = 0$.
2. **Multiplicativity:** $|xy| = |x| \cdot |y|$.
3. **Triangle Inequality:** $|x+y| \leq |x| + |y|$.

Proof. **(a)** Directly from the definition.

(b) If $x, y \geq 0$, the claim is trivial. In the general case, write $x = \pm x_0$, $y = \pm y_0$ with $x_0, y_0 \geq 0$. Then
$$|xy| = |\pm x_0 y_0| = |x_0 y_0| = |x_0| \cdot |y_0| = |x| \cdot |y|.$$

(c) Since $x \leq |x|$ and $y \leq |y|$, we get
$$x + y \leq |x| + |y|.$$

Also, $-x \leq |x|$ and $-y \leq |y|$, hence
$$-(x+y) \leq |x| + |y|.$$

Together these imply $|x+y| \leq |x| + |y|$. ■

Further Properties

1. $|1| = 1$, $|-1| = 1$, and in general
$$|-x| = |x| \quad \text{for all } x.$$

2. For $x, y \in \mathbb{R}$ with $y \neq 0$,
$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

   Proof: Since $x = \frac{x}{y} \cdot y$, multiplicativity gives
$$|x| = \left|\tfrac{x}{y}\right| \cdot |y|.$$

   Dividing by $|y|$ yields the claim.

3. For all $x, y \in \mathbb{R}$,
$$|x - y| \geq |x| - |y|, \qquad |x + y| \geq |x| - |y|.$$

   Proof: Write $x = (x - y) + y$. By the triangle inequality,
$$|x| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|.$$

   Replacing y with $-y$ gives
$$|x| - |y| \leq |x + y|.$$

Example

Let $f(x) = 2x^2 - 3x + 7$, with $x \in [-2, 2]$. Find a number $M > 0$ such that
$$|f(x)| \leq M \quad \text{for all } x \in [-2, 2].$$

Solution. Clearly, if $-2 \leq x \leq 2$, then $|x| \leq 2$. Applying the triangle inequality and the properties of the absolute value:
$$|f(x)| = |2x^2 - 3x + 7| \leq |2x^2| + |-3x| + |7| = 2|x|^2 + 3|x| + 7.$$

Since $|x| \leq 2$, we obtain
$$|f(x)| \leq 2(2)^2 + 3(2) + 7 = 21.$$

Conclusion. Thus, we can choose $M = 21$, and then
$$|f(x)| \leq 21 \quad \text{for all } x \in [-2, 2].$$

The Archimedean Axiom

Axiom 2 (Archimedean Axiom) *For any two real numbers $x, y > 0$ there exists a natural number $n \in \mathbb{N}$ such that*
$$nx > y.$$

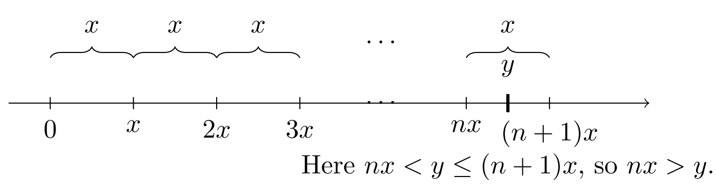


Figure 1.1: Illustration of the Archimedean axiom: multiples of x eventually exceed y.

Consequences of the Archimedean Axiom

Floor and Ceiling

Proposition 3 (Existence of Integer Bounds) *For every real number $x \in \mathbb{R}$ there exist natural numbers n_1, n_2 such that*

$$n_1 > x \quad \text{and} \quad -n_2 < x.$$

It follows that for each $x \in \mathbb{R}$ there exists a uniquely determined integer $n \in \mathbb{Z}$ with

$$n \le x < n+1.$$

This integer is denoted by $\lfloor x \rfloor$, the floor *of x. Instead of $\lfloor x \rfloor$, the notation $[x]$ (Gauss bracket) is also common.*
Similarly, there exists a unique integer $m \in \mathbb{Z}$ with

$$m-1 < x \le m,$$

which is denoted by $\lceil x \rceil$, the ceiling *of x.*

Archimedean Property with ε

Proposition 4 *For every $\varepsilon > 0$ there exists a natural number $n > 0$ such that*

$$\frac{1}{n} < \varepsilon.$$

Proof. There exists $n \in \mathbb{N}$ with $n > \frac{1}{\varepsilon}$. It follows that

$$\frac{1}{n} < \varepsilon.$$

■

Density of the Rationals

Theorem 5 (Density of $\mathbb{Q}$ in $\mathbb{R}$) *For any real numbers $a < b$ there exists a rational number $\dfrac{m}{n} \in \mathbb{Q}$ with*

$$a \ < \ \frac{m}{n} \ < \ b.$$

One-line proof. Choose $n \in \mathbb{N}$ so large that $\frac{1}{n} < b - a$ (Archimedean property). Let $m := \lfloor na \rfloor + 1$. Then $m - 1 \le na < m$, so $a < \frac{m}{n}$. Also

$$\frac{m}{n} \le \frac{na+1}{n} \ = \ a + \frac{1}{n} \ < \ a + (b-a) \ = \ b.$$

Thus $a < \frac{m}{n} < b$. ■

Corollary 6 (Density of $\mathbb{R} \setminus \mathbb{Q}$) *For any real numbers $a < b$ there exists an irrational number x with*

$$a < x < b.$$

Proof. Fix any irrational α (e.g. $\sqrt{2}$). By density of $\mathbb{Q}$, choose $q \in \mathbb{Q}$ with $a - \alpha < q < b - \alpha$. Then $x := q + \alpha$ satisfies $a < x < b$, and x is irrational (rational + irrational = irrational). ■

Mathematical Induction

Let n_0 be an integer and let $A(n)$ be a statement defined for each integer $n \geq n_0$. To prove that $A(n)$ holds for all $n \geq n_0$, it suffices to show:

(I.0) $A(n_0)$ is true. (Base step)

(I.1) For an arbitrary $n \geq n_0$, if $A(n)$ is true, then $A(n+1)$ is also true. (Induction step)

The idea of this proof principle is easy to see: From (I.0) we know that $A(n_0)$ is true. Applying (I.1) with $n = n_0$ yields the truth of $A(n_0+1)$. Applying (I.1) again gives $A(n_0+2)$, then $A(n_0+3)$, and so on. Thus $A(n)$ is valid for all $n \geq n_0$.

Example: Sum of the First n Natural Numbers

Theorem 7 *For every natural number n we have*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. Let $S(n) = 1 + 2 + \cdots + n$. We prove the formula

$$S(n) = \frac{n(n+1)}{2}$$

by induction.

Base step ($n = 1$). We have $S(1) = 1$ and

$$\frac{1(1+1)}{2} = 1,$$

so the formula holds for $n = 1$.

Induction step ($n \to n+1$). Assume $S(n) = \frac{n(n+1)}{2}$ holds (induction hypothesis). Then

$$S(n+1) = S(n) + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

Simplifying gives

$$S(n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Thus the formula also holds for $n+1$. ■

Bernoulli's Inequality

Theorem 8 (Bernoulli's Inequality) *Let $x \geq -1$. Then for all $n \in \mathbb{N}$,*

$$(1+x)^n \geq 1 + nx.$$

Proof. Base case ($n = 0$): Clearly $(1+x)^0 = 1 \geq 1$.

Induction step: Assume $(1+x)^n \geq 1 + nx$. Since $1 + x \geq 0$, multiplication gives

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

Thus the inequality holds for $n+1$. By induction, the statement holds for all n. ■

Growth of Powers

Theorem 9 (Growth of Powers) *Let $b > 0$ be a real number.*

1. *If $b > 1$, then for every $K \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that*
$$b^n > K.$$
2. *If $0 < b < 1$, then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that*
$$b^n < \varepsilon.$$

Proof. a) Let $x = b - 1 > 0$. By Bernoulli's inequality,

$$b^n = (1+x)^n \geq 1 + nx.$$

By the Archimedean axiom, there exists n such that $nx > K - 1$. For this n, we have $b^n > K$.

b) Since $b_1 := \frac{1}{b} > 1$, part (a) with $K := \frac{1}{\varepsilon}$ gives some n such that

$$b_1^n > \tfrac{1}{\varepsilon}.$$

This means

$$\frac{1}{b^n} > \frac{1}{\varepsilon},$$

and hence, by inversion of inequalities,

$$b^n < \varepsilon.$$

■

Binomial Coefficients and the Binomial Theorem

For natural numbers n and k, the *binomial coefficient* is defined as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdot \cdots \cdot k}.$$

Special cases:

$$\binom{n}{0} = 1, \qquad \binom{n}{1} = n, \qquad \binom{n}{k} = 0 \text{ if } k > n.$$

Another formula is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0 \leq k \leq n.$$

$$\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10, \qquad \binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20.$$

Lemma 10 (Pascal's Rule) *For all $n \geq 1$ and k,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This means every entry in Pascal's triangle is the sum of the two numbers above it.

Theorem 11 (Combinatorial Meaning) *$\binom{n}{k}$ counts the number of ways to choose k elements from an n-element set. In other words: $\binom{n}{k}$ is the number of different k-element subsets of an n-element set.*

Example 12 *The number of lottery tickets in "6 out of 49" is*

$$\binom{49}{6} = 13,983,816.$$

So the chance of guessing all 6 correctly is about 1 *in* 14 *million.*

Pascal's Triangle

The first rows are:

$$\begin{array}{lllll} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

Binomial Theorem

Theorem 13 *For real numbers x, y and natural number n:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Example 14 *For small n:*

$$\begin{aligned} (x+y)^0 &= 1, \\ (x+y)^1 &= x+y, \\ (x+y)^2 &= x^2+2xy+y^2, \\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3, \\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4. \end{aligned}$$

Useful Formulas

From the binomial theorem we also get:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Example 15 *For $n = 3$:*

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1+3+3+1 = 8 = 2^3.$$

Supremum and Infimum of Subsets of $\mathbb{R}$

Upper/Lower Bounds and Boundedness

Definition 16 *Let $A \subset \mathbb{R}$ be nonempty.*

- *An element $U \in \mathbb{R}$ is an* upper bound *of A if $x \leq U$ for all $x \in A$. We then say A is* bounded above *(equivalently, $A \subset (-\infty, U]$).*

- *An element* $L \in \mathbb{R}$ *is a* lower bound *of* A *if* $L \leq x$ *for all* $x \in A$. *We then say* A *is* bounded below *(equivalently,* $[L, \infty) \supset A$*).*
- *The set* A *is* bounded *if it is bounded above and below (equivalently,* $\exists M \geq 0$ *such that* $|x| \leq M$ *for all* $x \in A$*).*

Supremum and Infimum

Definition 17 (Least/greatest bounds) *Let $A \subset \mathbb{R}$ be nonempty.*

- $\alpha \in \mathbb{R}$ *is the* supremum *of* A *(least upper bound), written* $\alpha = \sup A$*, if*
  - *i)* $x \le \alpha$ *for all* $x \in A$ *(upper bound), and*
  - *ii) for every upper bound* U *of* A *we have* $\alpha \le U$ *(leastness).*
- $\beta \in \mathbb{R}$ *is the* infimum *of* A *(greatest lower bound), written* $\beta = \inf A$*, if*
  - *i)* $\beta \le x$ *for all* $x \in A$ *(lower bound), and*
  - *ii) for every lower bound* L *of* A *we have* $L \le \beta$ *(greatestness).*

Proposition 18 (Epsilon characterizations) *Let $A \subset \mathbb{R}$ be nonempty and bounded above/below.*

a) $\alpha = \sup A$ *iff*

$$\textit{(i) } x \le \alpha \ \forall x \in A \quad \textit{and} \quad \textit{(ii) } \forall \varepsilon > 0 \ \exists x_\varepsilon \in A : \ \alpha - \varepsilon < x_\varepsilon \le \alpha.$$

b) $\beta = \inf A$ *iff*

$$\textit{(i) } \beta \le x \ \forall x \in A \quad \textit{and} \quad \textit{(ii) } \forall \varepsilon > 0 \ \exists y_\varepsilon \in A : \ \beta \le y_\varepsilon < \beta + \varepsilon.$$

Proof. We prove (a); (b) is analogous. ($\Rightarrow$) If $\alpha = \sup A$, then (i) holds by definition. If (ii) failed, there would be $\varepsilon_0 > 0$ such that $x \le \alpha - \varepsilon_0$ for all $x \in A$, so $\alpha - \varepsilon_0$ would be an upper bound—contradiction to leastness. ($\Leftarrow$) Assume (i)–(ii). Let U be any upper bound of A. From (ii), $\alpha - \varepsilon < x_\varepsilon \le U$ for all $\varepsilon > 0$, hence $\alpha \le U$. Thus α is the least upper bound. ■

Proposition 19 (Uniqueness) *If* $\sup A$ *(resp.* $\inf A$*) exists, then it is unique.*

Proof. If α, α' are least upper bounds, then $\alpha \le \alpha'$ and $\alpha' \le \alpha$, hence $\alpha = \alpha'$. ■

Theorem 20 (Completeness of $\mathbb{R}$) *Every nonempty subset $A \subset \mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. Dually, every nonempty subset bounded below has an infimum.*

Approximating sup and inf by Sequences

Proposition 21 (Sequence characterization) *Let $A \subset \mathbb{R}$ be nonempty and bounded above/below.*

a) If $\alpha = \sup A$*, there exists a sequence* $(x_n) \subset A$ *with* $x_n \uparrow \alpha$ *and* $x_n \to \alpha$*.*

b) If $\beta = \inf A$*, there exists a sequence* $(y_n) \subset A$ *with* $y_n \downarrow \beta$ *and* $y_n \to \beta$*.*

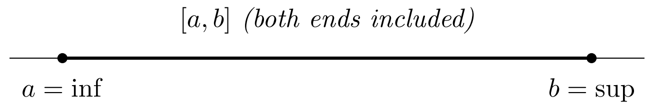
Proof. (a) By Prop. 18, choose $x_n \in A$ with $\alpha - \frac{1}{n} < x_n \le \alpha$. Then $x_n \to \alpha$ and one may choose a nondecreasing subsequence. The proof of (b) is analogous. ■

Examples

Example 22 (Closed interval: max/min attained) *For the closed interval $[a, b]$ with $a \le b$,*

$$\sup[a, b] = b \quad \textit{and} \quad \inf[a, b] = a.$$

Easy idea. *Every point of* $[a,b]$ *lies* at or to the left of b*, so* b *is an upper bound. No smaller number can be an upper bound because* $b \in [a,b]$*. Thus* b *is the smallest upper bound (the supremum). The argument for* $\inf[a,b] = a$ *is the same, using the left endpoint.* ■



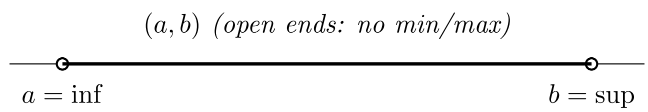
Example 23 (Open interval: barriers not attained) *For the open interval* (a,b) *with* $a < b$*,*

$$\sup(a,b) = b \quad \text{and} \quad \inf(a,b) = a.$$

Why b ***is the least upper bound.*** *(1)* Upper bound: *Every* $x \in (a,b)$ *satisfies* $x < b$*, so* b *is an upper bound.*
(2) Leastness: *Let* K *be any upper bound of* (a,b)*. For any* $\varepsilon > 0$*:*

- *If* $\varepsilon \le b - a$*, then* $x = b - \varepsilon/2$ *belongs to* (a,b) *and we have* $b - \varepsilon < x \le K$*, hence* $b \le K + \varepsilon$*. Since* ε *is arbitrary,* $b \le K$*.*
- *If* $\varepsilon > b - a$*, then* $b - \varepsilon \le a$*, and* any $x \in (a,b)$ *satisfies* $b - \varepsilon < x \le K$*, so again* $b \le K + \varepsilon$ *and thus* $b \le K$*.*

Therefore no number $< b$ *can be an upper bound;* b *is the least upper bound. The proof for* $\inf(a,b) = a$ *is analogous.* ■



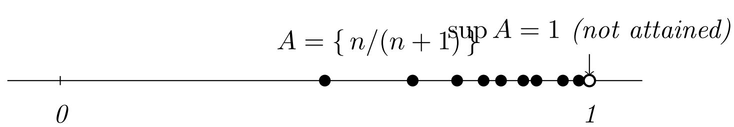
Example 24 (A discrete set approaching a limit) *Let*

$$A := \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}.$$

Then $\sup A = 1$*, but* $1 \notin A$ *(so no maximum).* ***Easy check.*** *Upper bound:* $\frac{n}{n+1} < 1$ *for all* n*, so* 1 *is an upper bound. Sharpness: Given any* $\varepsilon > 0$*, choose* n *so large that* $\frac{1}{n+1} < \varepsilon$ *(Archimedean property). Then*

$$\frac{n}{n+1} \;=\; 1 - \frac{1}{n+1} \;>\; 1 - \varepsilon,$$

so no number smaller than 1 *can bound* A *from above. Hence* $\sup A = 1$*.* ■



Example 25 (Competing growth: n^2 **vs.** 2^n**)** *Let*

$$A := \left\{ \frac{n^2}{2^n} : n \in \mathbb{N} \right\}.$$

Then

$$\sup A \;=\; \max A \;=\; \frac{9}{8} \quad \text{attained at } n = 3.$$

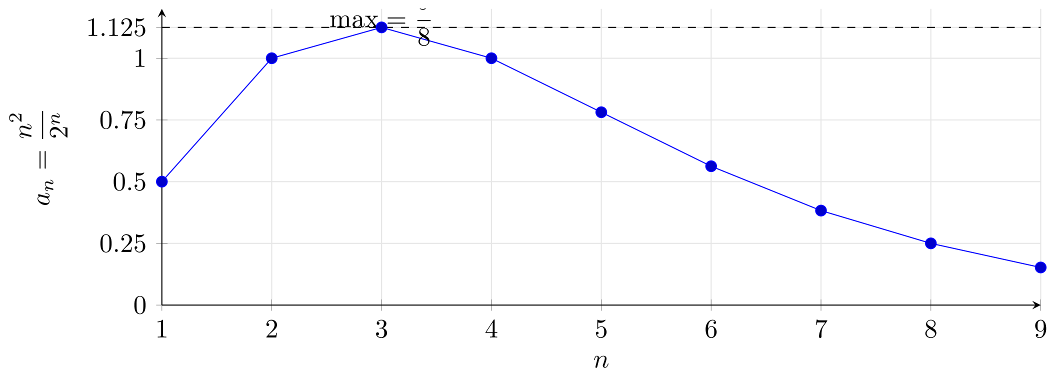
Why the maximum is at $n = 3$***.*** *Compute a few terms:*

$$\frac{1}{2}, \quad 1, \quad \frac{9}{8}, \quad 1, \quad \frac{25}{32}, \ldots$$

The sequence rises up to $n = 3$ and then decreases. Formally, for $n \geq 3$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{(n+1)^2}{2n^2} \leq 1 \quad \iff \quad (n+1)^2 \leq 2n^2 \iff n^2 - 2n - 1 \geq 0,$$

which holds for all $n \geq 3$. Thus (a_n) is decreasing for $n \geq 3$, and since $a_3 = \frac{9}{8}$ and $a_2 = 1 < a_3$, the maximum value is $a_3 = \frac{9}{8}$. ■



How to find sup and inf

1. **Picture the set on the number line.** What is the tightest left/right barrier?
2. **Upper bound test:** Show every $x \in A$ is $\leq$ your candidate α.
3. **Sharpness test (epsilon idea):** For every $\varepsilon > 0$, find $x_\varepsilon \in A$ with $\alpha - \varepsilon < x_\varepsilon \leq \alpha$. This proves $\alpha = \sup A$.
4. **Same for inf:** Lower bound + sharpness: for all $\varepsilon > 0$ find $y_\varepsilon \in A$ with $\beta \leq y_\varepsilon < \beta + \varepsilon$.
5. **Check attainment:** If $\sup A \in A$, it is the maximum; if $\inf A \in A$, it is the minimum.

Chapter 2

Sequences and Limits

A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$, $n \mapsto a_n$. We write $(a_n)_{n\in\mathbb{N}}$, or briefly (a_n), or $(a_0, a_1, a_2, \dots)$. More generally one may start at any integer k, writing $(a_n)_{n\geq k}$.

Examples

(1) Constant sequence: $a_n = a$ for all n gives $(a, a, a, \dots)$.

(2) Harmonic tail: $a_n = \frac{1}{n}$ for $n \geq 1$ gives $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$.

(3) Alternating signs: $a_n = (-1)^n$ gives $(+1, -1, +1, -1, \dots)$.

(4) Ratios: $a_n = \frac{n}{n+1}$ gives $\left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$.

(5) Powers: $(x^n)_{n\in\mathbb{N}} = (1, x, x^2, x^3, \dots)$ for a fixed $x \in \mathbb{R}$.

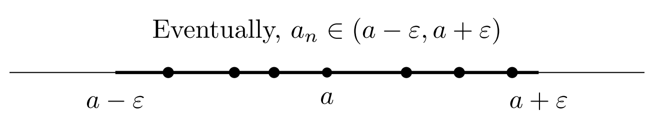
Definition of limit (epsilon–N)

Definition 26 (Convergence) *A sequence* (a_n) converges *to* $a \in \mathbb{R}$ *if*

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \text{ such that } |a_n - a| < \varepsilon \quad \text{for all } n \geq N.$$

We write $\lim\limits_{n\to\infty} a_n = a$, *or* $a_n \to a$ *as* $n \to \infty$. *If a sequence does not converge, it is called* divergent.

For $\varepsilon > 0$, the *ε–neighborhood* of a is the open interval $(a - \varepsilon, a + \varepsilon)$. Convergence $a_n \to a$ means: *eventually* (i.e., for all sufficiently large n) every term a_n lies inside $(a - \varepsilon, a + \varepsilon)$, no matter how small ε is.



Basic examples

Example 27 ***(1)*** *If* $a_n = a$ *for all* n, *then* $\lim\limits_{n\to\infty} a_n = a$.

Let $\varepsilon > 0$ *be arbitrary. Take any index* N *(e.g.,* $N = 1$ *if the sequence starts at* 1*). For all* $n \geq N$,

$$|a_n - a| = |a - a| = 0 < \varepsilon.$$

By the ε–N *definition of limit,* $\lim_{n\to\infty} a_n = a$.

(2) $\lim_{n\to\infty} \frac{1}{n} = 0$ *Given* $\varepsilon > 0$, *by the Archimedean property choose* N *with* $N > \frac{1}{\varepsilon}$. *Then for all* $n \geq N$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$.

(3) *The sequence* $a_n = (-1)^n$ diverges. *If it converged to* a, *then for* $\varepsilon = 1$ *all large* n *would satisfy* $|a_n - a| < 1$ *and* $|a_{n+1} - a| < 1$, *which implies* $|a_{n+1} - a_n| \leq |a_{n+1} - a| + |a - a_n| < 2$, *contradicting* $|(-1)^{n+1} - (-1)^n| = 2$.

(4) $\lim_{n\to\infty} \frac{n}{n+1} = 1$. *Since* $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1}$, *the same choice* $N > \frac{1}{\varepsilon}$ *works.*

(5) $\lim_{n\to\infty} \frac{n}{2^n} = 0$. *For* $n \geq 4$ *one has* $n^2 \leq 2^n$ *(e.g. by induction), hence* $0 \leq \frac{n}{2^n} \leq \frac{1}{n}$ *for* $n \geq 4$. *Since* $\frac{1}{n} \to 0$, *the squeeze gives* $\frac{n}{2^n} \to 0$.

Boundedness

Definition 28 (Bounded sequences) *A sequence* (a_n) *is* bounded *if* $\exists M \geq 0$ *such that* $|a_n| \leq M$ *for all* n. *It is bounded above (resp. below) if* $a_n \leq K$ *(resp.* $a_n \geq K$*) for all* n *for some* $K \in \mathbb{R}$.

Theorem 29 (Every convergent sequence is bounded) *If* $a_n \to a$, *then* (a_n) *is bounded.*

Proof. Choose N with $|a_n - a| < 1$ for all $n \geq N$. Then $|a_n| \leq |a| + 1$ for $n \geq N$. Let $M := \max\{|a_0|, \ldots, |a_{N-1}|, |a| + 1\}$. Then $|a_n| \leq M$ for all n. ■

Remark

The converse is false: $a_n = (-1)^n$ is bounded but not convergent.

Powers x^n

Proposition 30 *Let* $x \in \mathbb{R}$. *Then:*

a) *If* $|x| < 1$, *then* $x^n \to 0$.

b) *If* $x = 1$, *then* $x^n = 1$ *for all* n *(converges to* 1*).*

c) *If* $x = -1$, *then* $x^n = (-1)^n$ *diverges.*

d) *If* $|x| > 1$, *then* (x^n) *is unbounded and hence diverges.*

Idea. (a) Given $\varepsilon > 0$, use the Archimedean property to find N with $|x|^N < \varepsilon$; then $|x|^n \leq |x|^N < \varepsilon$ for all $n \geq N$. The other cases are immediate. ■

Uniqueness and algebra of limits

Theorem 31 (Uniqueness) *If* $a_n \to a$ *and* $a_n \to b$, *then* $a = b$.

Proof. If $a \neq b$, set $\varepsilon = \frac{|a-b|}{2}$. For large n we have both $|a_n - a| < \varepsilon$ and $|a_n - b| < \varepsilon$, so $|a - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon = |a - b|$, a contradiction. ■

Theorem 32 (Sum and product) *If* $a_n \to a$ *and* $b_n \to b$, *then* $(a_n + b_n) \to a + b$ *and* $(a_n b_n) \to ab$.

Sketch. For sums: $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$. For products: write $a_n b_n - ab = a_n(b_n - b) + (a_n - a)b$ and use boundedness of (a_n). ■

Corollary 33 (Linear combinations) *If $a_n \to a$ and $b_n \to b$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda a_n + \mu b_n \to \lambda a + \mu b$. In particular, a_n and b_n have the same limit iff $(a_n - b_n)$ is a null sequence.*

Theorem 34 (Quotients) *If $a_n \to a$ and $b_n \to b \neq 0$, then $b_n \neq 0$ for all large n and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Idea. First show $\frac{1}{b_n} \to \frac{1}{b}$ using that $|b_n - b| < \frac{|b|}{2}$ eventually, hence $|b_n| \geq \frac{|b|}{2}$. Then combine with (a_n) via the product rule. ■

Example 35 (A rational expression) *Let $a_n = \dfrac{3n^2 + 13n}{n^2 - 2}$ for $n \in \mathbb{N}$. Then*

$$a_n = \frac{3 + \frac{13}{n}}{1 - \frac{2}{n^2}} \xrightarrow[n\to\infty]{} \frac{3+0}{1-0} = 3.$$

Order and limits

Theorem 36 (Order preserved in the limit) *If $a_n \leq b_n$ for all n and $a_n \to a$, $b_n \to b$, then $a \leq b$.*

Corollary 37 (Closed interval trap) *If $A \leq a_n \leq B$ for all n and $a_n \to L$, then $A \leq L \leq B$.*

4. Examples

Example 1:

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3}$$

Highest powers dominate:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2}{3n^2} = \frac{2}{3}$$

Justification: Divide numerator and denominator by n^2:

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3} = \frac{2 - \frac{3}{n}}{3 + \frac{5}{n} + \frac{3}{n^2}}.$$

Since

$$\frac{1}{n} \to 0 \quad \text{and} \quad \frac{1}{n^2} \to 0 \quad \text{as } n \to \infty,$$

we get:

$$\lim_{n\to\infty} a_n = \frac{2-0}{3+0+0} = \frac{2}{3}.$$

General Rule: For $a_n = \frac{P(n)}{Q(n)}$:

- If $\deg P < \deg Q$, then $\lim a_n = 0$
- If $\deg P = \deg Q$, then $\lim a_n$ equals the ratio of leading coefficients
- If $\deg P > \deg Q$, then $\lim a_n = \infty$ or $-\infty$ (divergent)

Example 2: Geometric and Power Sequences

Geometric sequence:

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1, \\ 1, & \text{if } r = 1, \\ \text{diverges}, & \text{if } |r| \geq 1 \text{ and } r \neq 1. \end{cases}$$

We can write:

$$r^n = e^{n \ln r}, \quad \text{for } r > 0.$$

- **If** $0 < r < 1$: then $\ln r < 0$ and $n \ln r \to -\infty$.

$$\Rightarrow \quad r^n = e^{n \ln r} \to 0.$$

- **If** $r = 1$: then $\ln r = 0$ and

$$\Rightarrow \quad r^n = e^0 = 1.$$

- **If** $r > 1$: then $\ln r > 0$ and $n \ln r \to \infty$.

$$\Rightarrow \quad r^n = e^{n \ln r} \to \infty.$$

- **If** $r < 0$: then r^n diverges and may oscillate, depending on the parity of n.

Power sequence (also called "dual geometric"):

$$\lim_{n\to\infty} n^r = \begin{cases} 0, & \text{if } r < 0, \\ 1, & \text{if } r = 0, \\ \infty, & \text{if } r > 0. \end{cases}$$

- $n^0 = 1$ for all n,
- $n^r \to \infty$ if $r > 0$,
- $n^r = \dfrac{1}{n^{-r}} \to 0$ if $r < 0$.

Example 3:

Show that

$$\lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right) = 1.$$

As

$$\frac{1}{n} \to 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

we rewrite:

$$n \cdot \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \to 1.$$

General Rule:

- If $\lim_{x\to 0} f(x) = L$, then $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = L$.
- If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

Example 4

If $\alpha > 0$, then:

$$\lim_{n\to\infty} \alpha^{1/n} = 1.$$

Let $a_n = \alpha^{1/n}$. Then:

$$\ln a_n = \frac{\ln \alpha}{n} \to 0.$$

So,

$$a_n = e^{\ln a_n} \to e^0 = 1.$$

Example 5

The sequence $\left(n^{1/n}\right)$ converges to 1:

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Let $a_n = n^{1/n}$. Then:

$$\ln a_n = \frac{\ln n}{n}.$$

To evaluate the limit, apply L'Hôpital's Rule to:

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

Since both numerator and denominator tend to ∞, we differentiate top and bottom:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Therefore:

$$\ln a_n \to 0 \quad \Rightarrow \quad a_n = e^{\ln a_n} \to e^0 = 1.$$

5. Monotone Sequences

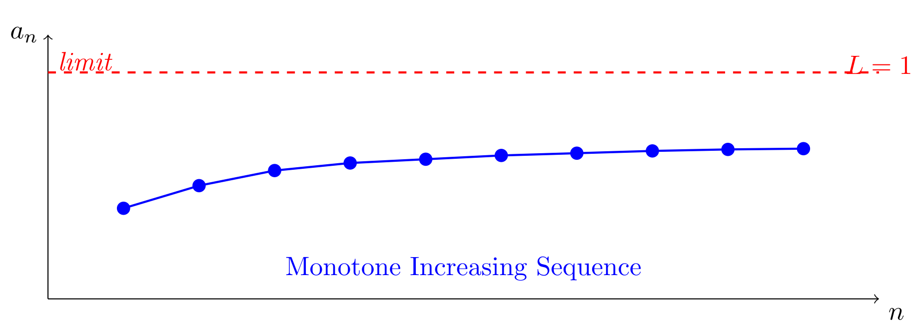
A sequence $\{a_n\}$ is:

- **Increasing** if $a_{n+1} \geq a_n$ for all n
- **Decreasing** if $a_{n+1} \leq a_n$ for all n
- **Monotonic** if it is either increasing or decreasing

Theorem (Monotone Convergence Theorem): If a sequence is monotonic and bounded, then it converges.

Example 1:

Let $a_n = 1 - \frac{1}{n}$. It is increasing and bounded above by 1, so the sequence a_n is convergent.



Explanation:

- The blue points represent terms a_n of a sequence.
- Each term is greater than or equal to the previous — the sequence is increasing.
- The red dashed line is the horizontal asymptote at $L = 1$, showing the limit.
- The sequence approaches the limit from below but never exceeds it.

Example 2: Sequence of partial sum

Let a_n be a sequence of non-negative numbers, meaning that $a_n \geq 0$ for every $n \in \mathbb{N}$. The sequence of partial sums associated with a_n is defined by:

$$S_N = \sum_{k=1}^{N} a_k = a_1 + a_2 + \cdots + a_N.$$

This means that :

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Since each term $a_n \geq 0$, adding a new term always makes the total sum stay the same or increase:

$$S_{N+1} - S_N = a_{N+1} \geq 0.$$

We now consider two possible situations:

- **Case 1:** The sequence (S_N) is bounded above.
  Since S_N is increasing and bounded, the Monotonic Convergence Theorem tells us that S_N converges to a finite limit.
- **Case 2:** The sequence (S_N) is not bounded above.
  In this case, the partial sums grow without limit, that is,

$$\lim_{N \to \infty} S_N = \infty.$$

6. Squeeze Theorem

Theorem 38 *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that:*

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N,$$

and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then:

$$\lim_{n \to \infty} b_n = L.$$

Example 1:

Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Since $-1 \le \sin n \le 1$, we have:

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

Both bounds go to 0, so by the Squeeze Theorem:

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Sequences Divergence to $\pm\infty$

Definition 39 (Sequences divergence to $\pm\infty$) *A real sequence $(a_n)_{n\in\mathbb{N}}$* diverges to $+\infty$ *if*

$$\forall K \in \mathbb{R} \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; a_n > K \quad \text{for all } n \ge N.$$

It diverges to $-\infty$ *if*

$$\forall K \in \mathbb{R} \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; a_n < K \quad \text{for all } n \ge N,$$

equivalently, if $(-a_n)$ diverges to $+\infty$. We write $\lim\limits_{n\to\infty} a_n = +\infty$ or $\lim\limits_{n\to\infty} a_n = -\infty$. Some authors say the sequence converges improperly *(or* in the extended reals*) to $\pm\infty$.*

Example 40 *(1) $a_n = n$ diverges to $+\infty$.*

(2) $a_n = -2n$ diverges to $-\infty$.

(3) $a_n = (-1)^n n$ is divergent but neither to $+\infty$ nor to $-\infty$ (it is unbounded in both directions).

Remark 41 *(a) If $a_n \to +\infty$ (resp. $-\infty$), then (a_n) is not bounded above (resp. not bounded below). The converse need not hold, as $a_n = (-1)^n n$ shows.*

(b) The symbols $+\infty$ and $-\infty$ are not real numbers*. They record limiting behavior; one must not manipulate them as if they were reals. For instance, from $\lim a_n = \infty$, $\lim b_n = 1$, and $\lim(a_n + b_n) = \infty$, it is* not *legitimate to deduce the real equality $\infty+1 = \infty$ (that would lead to contradictions like $1 = 0$). It is sometimes convenient to enlarge $\mathbb{R}$ to the* extended real line *$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ with order $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.*

Theorem 42 (Reciprocal of a sequence diverging to $\pm\infty$) *If $a_n \to +\infty$ or $a_n \to -\infty$, then there exists n_0 such that $a_n \neq 0$ for all $n \ge n_0$ and*

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof. Assume $a_n \to +\infty$. Taking $K = 0$ in the definition yields $a_n > 0$ for all $n \ge n_0$, hence $a_n \neq 0$ eventually. Given $\varepsilon > 0$, choose N so that $a_n > \frac{1}{\varepsilon}$ for all $n \ge N$; then $\left|\frac{1}{a_n} - 0\right| < \varepsilon$ for $n \ge N$. The case $a_n \to -\infty$ reduces to the previous one by applying the result to $-a_n$. ■

Theorem 43 (Reciprocal of a null sequence with fixed sign) *Let (a_n) be a null sequence, $a_n \to 0$.*

a) If $a_n > 0$ for all sufficiently large n, then $\dfrac{1}{a_n} \to +\infty$.

b) If $a_n < 0$ for all sufficiently large n, then $\dfrac{1}{a_n} \to -\infty$.

Proof. We prove (a); (b) is analogous. Let $K > 0$. Since $a_n \to 0$, there is N with $0 < a_n < \frac{1}{K}$ for all $n \geq N$. Then $\frac{1}{a_n} > K$ for $n \geq N$, so $\frac{1}{a_n} \to +\infty$. ■

Example 44 *From* $\lim_{n\to\infty} \dfrac{n}{2^n} = 0$ *(e.g. by induction* $n^2 \leq 2^n$ *for* $n \geq 4$ *and a squeeze), Theorem 43 gives*

$$\lim_{n\to\infty} \frac{2^n}{n} = +\infty.$$

Chapter 3

Infinite series

1. Introduction

An **infinite series** is the sum of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

To analyze the convergence of this series, we consider the **sequence of partial sums**:

$$S_N = \sum_{k=1}^{N} a_k.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** if the limit of the sequence $\{S_N\}$ exists and is finite. In that case, the value of the infinite sum is defined as:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If this limit does not exist or is infinite, the series is said to **diverge**.

2. Telescoping Series

Evaluate the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We first decompose the general term using partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

and then consider the partial sum:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

We expand the partial sum:

$$S_N = \left(\frac{\mathbf{1}}{\mathbf{1}} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{\mathbf{1}}{\mathbf{N+1}}\right).$$

After cancellation, only the first and the last term remain:

$$S_N = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{N+1}}.$$

Taking the limit as $N \to \infty$, we find the sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

General Rule (Telescoping Form):
If a sequence satisfies $a_n = b_n - b_{n+1}$, then the partial sum telescopes:

$$S_N = \sum_{n=1}^{N} a_n = b_1 - b_{N+1} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = b_1 - \lim_{N\to\infty} b_{N+1},$$

provided the limit exists.

Application to the Geometric Series $\sum_{n=0}^{\infty} r^n$:

$$\begin{aligned}(1-r)\sum_{n=0}^{N} r^n &= \sum_{n=0}^{N} \left(r^n - r^{n+1}\right) \\ &= r^0 - r^{N+1} = 1 - r^{N+1}.\end{aligned}$$

Dividing both sides by $1-r$, we obtain the formula for the partial sum:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad r \neq 1.$$

Geometric Series Convergence ("r-Test"):

$$\sum_{n=0}^{\infty} r^n \text{ converges} \quad \Longleftrightarrow \quad |r| < 1.$$

Sum Formula:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1, \\ \text{diverges}, & \text{if } |r| \geq 1. \end{cases}$$

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$$

3. Basic Properties

- **Linearity:**
$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
- **Multiplication by a constant:**
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$
- **Necessary condition for convergence:**
If $\sum_{n=1}^{\infty} a_n$ converges, then it must be that $a_n \to 0$ as $n \to \infty$. However, the converse is not true: the fact that $a_n \to 0$ does not guarantee convergence of the series.

  Counterexample: Let $a_n = \frac{1}{n}$. Then $a_n \to 0$, but the harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
  diverges.

Justification: The necessary condition for convergence follows from the identity:

$$S_N = \sum_{k=1}^{N} a_k \quad \Longleftrightarrow \quad a_n = S_n - S_{n-1}, \quad \text{with } S_0 = 0,$$

and the fact that if $\sum a_n$ converges, then both S_n and S_{n-1} converge to the same limit, which forces $a_n \to 0$.

4. Convergence Tests for Positive Series

We are concerned with series $\sum_{n=1}^{\infty} a_n$, such that $a_n \geq 0$ for all n (positive series).

(a) nth-Term Test

If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

(b) Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges. If $\sum a_n$ diverges and $b_n \geq a_n \geq 0$, then $\sum b_n$ also diverges.

(c) Limit Comparison Test

Let $a_n, b_n > 0$. If:
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$
then either both series converge or both diverge.

(d) Ratio Test

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

- If $L < 1$: the series converges.
- If $L > 1$ or $L = \infty$: the series diverges.
- If $L = 1$: the test is inconclusive.

(e) Root Test

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}, \quad \text{same conclusions as the Ratio Test}$$

(f) Integral Test

Let $f : [1, \infty) \to \mathbb{R}$ be a function such that:

- f is continuous,
- $f(x) \geq 0$ for all $x \geq 1$,
- f is decreasing on $[1, \infty)$,
- $f(n) = a_n$ for all integers $n \geq 1$.

Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

Application: The p-Series Test
Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } p > 0.$$

Let $f(x) = \frac{1}{x^p}$, which is continuous, positive, and decreasing for $x \geq 1$ when $p > 0$. We apply the integral test by evaluating

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx.$$

Case 1: $p \neq 1$

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

- If $p > 1$, then $1 - p < 0$, so $t^{1-p} \to 0$ as $t \to \infty$, and the integral converges to $\frac{1}{p-1}$.
- If $p < 1$, then $1 - p > 0$, so $t^{1-p} \to \infty$, and the integral diverges.

Case 2: $p = 1$

$$\int_1^t \frac{1}{x}\,dx = \ln t \quad \to \infty \quad \text{as } t \to \infty.$$

So the integral diverges.

Conclusion:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

By the integral test, the same result holds for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

Examples:

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$: converges (since $p = 2 > 1$)

- $\sum_{n=1}^{\infty} \frac{1}{n}$: diverges (harmonic series, $p = 1$)
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: diverges (since $p = \frac{1}{2} < 1$)

6. Absolute and Conditional Convergence

- A series $\sum a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges.
- A series $\sum a_n$ is said to be **conditionally convergent** if $\sum a_n$ converges, but the series $\sum |a_n|$ diverges.

Theorem (Absolute Convergence Implies Convergence): If the series $\sum |a_n|$ converges, then the original series $\sum a_n$ also converges.

The converse is not true. That is, a convergent series $\sum a_n$ does not necessarily imply that $\sum |a_n|$ converges. A classical example is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

which converges conditionally, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (harmonic series).

(g) Alternating Series Test (Leibniz Test)

If $a_n > 0$, decreasing, and $\lim a_n = 0$, then:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges}$$

- $\sum \frac{(-1)^n}{n^2}$: absolutely convergent (use p-series test)
- $\sum \frac{(-1)^n}{\ln n}$: conditionally convergent (use Leibniz test)
- $\sum \frac{n!}{n^n}$: converges (use ratio or root test)

Chapter 4

The Riemann Integral

1. Partition and Refinement of an Interval

Let $[a, b]$ be a closed and bounded interval with $a < b$. A **partition** P of $[a, b]$ is a finite ordered set of points

$$P = \{x_0, x_1, \dots, x_n\}, \qquad a = x_0 < x_1 < \dots < x_n = b,$$

which subdivides $[a, b]$ into the n subintervals

$$[x_{k-1}, x_k], \qquad k = 1, 2, \dots, n.$$

These subintervals are pairwise disjoint in their interiors and their union is $[a, b]$.

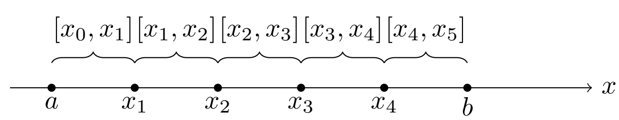


Figure 4.1: Partition P of $[a, b]$ into subintervals.

Let

$$P = \{x_0, \dots, x_n\} \quad \text{with} \quad a = x_0 < \dots < x_n = b.$$

A partition Q of $[a, b]$ is called a **refinement** of P if $P \subseteq Q$; that is, every point of P also appears in Q, and Q may contain additional points inside the subintervals determined by P.

Example

Suppose

$$P = \{a, x_1, x_2, b\}, \qquad a < x_1 < x_2 < b,$$

and we insert three additional points

$$q_1 \in (a, x_1), \qquad q_2 \in (x_1, x_2), \qquad q_3 \in (x_2, b).$$

Then the refinement Q is

$$Q = P \cup \{q_1, q_2, q_3\} = \{a, q_1, x_1, q_2, x_2, q_3, b\},$$

listed in strictly increasing order.

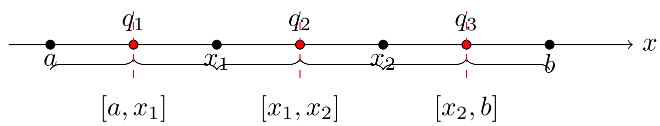


Figure 4.2: Refinement Q of P by inserting q_1, q_2, and q_3.

2. Lower and Upper Sums

Definition 45 (Lower and Upper Sums) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a, b]$. For each subinterval $[x_{k-1}, x_k]$, define:*

$$m_k := \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

*Then the **lower sum** of f with respect to P is:*

$$L(f, P) = \sum_{k=1}^{n} m_k \cdot (x_k - x_{k-1}),$$

*and the **upper sum** is:*

$$U(f, P) = \sum_{k=1}^{n} M_k \cdot (x_k - x_{k-1}).$$

3. Properties of Riemann Sums

Lemma 46 (Properties of Lower and Upper Sums) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then:*

1. *For every partition P,*
$$L(f, P) \le U(f, P).$$
2. *If Q is a refinement of P, then*
$$L(f, P) \le L(f, Q) \quad \text{and} \quad U(f, P) \ge U(f, Q).$$
3. *For any two partitions P_1, P_2,*
$$L(f, P_1) \le U(f, P_2).$$

Proof.

1. **Lower sum is always less than or equal to upper sum.**
For each subinterval $[x_{k-1}, x_k]$, we define:

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Since $m_k \le M_k$ for all k, it follows that:

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k = U(f, P).$$

Example: Let $f(x) = x^2$ on $[0, 1]$, and let $P = \{0, 0.5, 1\}$. Then:

$L(f, P) = 0^2 \cdot 0.5 + (0.5)^2 \cdot 0.5 = 0 + 0.125 = 0.125, U(f, P) = (0.5)^2 \cdot 0.5 + (1)^2 \cdot 0.5 = 0.125 + 0.5 = 0.625.$

So $L(f,P) < U(f,P)$.

2. **Refining increases lower sum and decreases upper sum.**
A refinement Q of P adds points to subdivide the interval more finely. The infimum over a smaller subinterval is at least as large as over the larger one (because we're minimizing over fewer values), and similarly, the supremum over a smaller subinterval is at most as large.

Hence:
$$L(f,Q) \geq L(f,P), \qquad U(f,Q) \leq U(f,P).$$

Example: Use the same $f(x) = x^2$ on $[0,1]$, but refine $P = \{0, 0.5, 1\}$ to $Q = \{0, 0.25, 0.5, 0.75, 1\}$. You will find:
$$L(f,Q) > L(f,P), \quad U(f,Q) < U(f,P).$$

3. **Lower sum of one partition is less than upper sum of another.**
Let $R = P_1 \cup P_2$, which is a common refinement of both P_1 and P_2. Then, by part (2):
$$L(f,P_1) \leq L(f,R) \leq U(f,R) \leq U(f,P_2),$$
so:
$$L(f,P_1) \leq U(f,P_2).$$

Example: Let $P_1 = \{0, 0.5, 1\}$, $P_2 = \{0, 0.25, 1\}$. Their union is $R = \{0, 0.25, 0.5, 1\}$. Again using $f(x) = x^2$, you can compute and verify the inequality numerically:
$$L(f,P_1) \leq L(f,R) \leq U(f,R) \leq U(f,P_2).$$

■

Definition 47 *A bounded function $f : [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** (or simply **integrable**) if its lower integral*
$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$
coincides with its upper integral
$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$
*The common value of $L(f)$ and $U(f)$ is called the **Riemann integral** of f over the interval $[a,b]$, and is denoted by*
$$\int_a^b f \quad \text{or more explicitly} \quad \int_a^b f(x)\,dx.$$

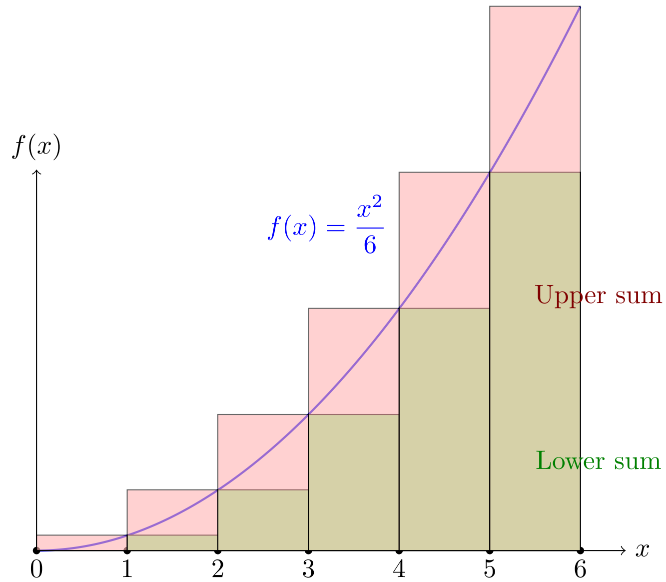


Figure 4.3: Lower and upper sums for the function $f(x) = \frac{x^2}{6}$ on $[0, 6]$.

Intuitively, a bounded function f is Riemann integrable if we can approximate the area under its graph from below (using lower sums) and from above (using upper sums) in such a way that both approximations can be made arbitrarily close to each other by refining the partition.

In the figure above:

- The **green rectangles** represent the *lower sum* $L(f, P)$, constructed using the minimum value of f on each subinterval.
- The **red translucent rectangles** represent the *upper sum* $U(f, P)$, constructed using the maximum value of f on each subinterval.
- The **blue curve** shows the graph of the function $f(x) = \frac{x^2}{6}$.

As the partition becomes finer (i.e., we divide $[a, b]$ into smaller subintervals), the lower and upper rectangles better approximate the area under the curve. The difference between the total areas of the upper and lower sums decreases.

This leads to the following fundamental characterization of Riemann integrability:

> A bounded function f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that:
> $$U(f, P) - L(f, P) < \varepsilon.$$

This ensures that all upper and lower sums are squeezed around a single unique value — the Riemann integral of the function.

Lemma 48 *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Example 49 (Direct applications of the lemma) *We illustrate the lemma with two explicit examples.*

(a) ***Constant function:*** *Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$, where $c \in \mathbb{R}$ is constant.*

Since f is constant, on every subinterval $[x_{k-1}, x_k]$ of any partition P, the infimum and supremum satisfy:

$$m_k = M_k = c.$$

Therefore, both the lower sum and the upper sum are equal:

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k = c(b - a), \quad U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k = c(b - a).$$

It follows that

$$U(f, P) - L(f, P) = 0 < \varepsilon \quad \text{for all } \varepsilon > 0,$$

so the lemma is satisfied trivially. Thus, f is Riemann integrable and its integral is:

$$\int_a^b f(x)\, dx = \int_a^b c\, dx = c(b - a).$$

(b) ***Quadratic function:*** *Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$.*

We construct a sequence of uniform partitions:

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}, \quad n \in \mathbb{N}.$$

Each subinterval has width $\Delta x = \frac{1}{n}$. On the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, the function $f(x) = x^2$ is increasing, so:

$$m_k = \left(\frac{k-1}{n}\right)^2, \quad M_k = \left(\frac{k}{n}\right)^2.$$

The lower and upper sums are:

$$L(f, P_n) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6},$$

$$U(f, P_n) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Therefore, the difference between the upper and lower sums is:

$$U(f, P_n) - L(f, P_n) = \frac{1}{n^3} \left(\frac{n(n+1)(2n+1) - (n-1)n(2n-1)}{6}\right).$$

This expression tends to 0 as $n \to \infty$, hence for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$U(f, P_n) - L(f, P_n) < \varepsilon.$$

By the lemma, $f(x) = x^2$ is Riemann integrable on $[0, 1]$, and we have:

$$\int_0^1 f(x)\, dx = \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n) = \frac{1}{3}.$$

Theorem 50 *Every monotone function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Suppose f is monotone increasing on $[a,b]$. Then f is bounded, since

$$f(a) \le f(x) \le f(b) \quad \text{for all } x \in [a,b].$$

Let $\varepsilon > 0$ be given. We want to find a partition P such that $U(f,P) - L(f,P) < \varepsilon$. Choose $\delta > 0$ such that

$$\delta(f(b) - f(a)) < \varepsilon.$$

Now select a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that the width of every subinterval satisfies:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

Since f is increasing, on each subinterval $[x_{k-1}, x_k]$ we have:

$$m_k = f(x_{k-1}), \quad M_k = f(x_k),$$

so the difference between the upper and lower sums becomes:

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}).$$

Using the fact that $x_k - x_{k-1} < \delta$, we estimate:

$$U(f,P) - L(f,P) \le \delta \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) < \varepsilon.$$

Hence, by the integrability criterion (Lemma), f is Riemann integrable. ■

Theorem 51 *Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Since f is continuous on the closed interval $[a,b]$, which is compact, the **Extreme Value Theorem** guarantees that f is bounded and attains its maximum and minimum on each subinterval of any partition. Furthermore, by the **Uniform Continuity Theorem**, f is uniformly continuous on $[a,b]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

On each subinterval $[x_{k-1}, x_k]$, the function f attains both its maximum M_k and minimum m_k (by continuity), and we have:

$$M_k - m_k < \frac{\varepsilon}{b-a}.$$

Thus,

$$U(f,P) - L(f,P) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Hence, by the integrability criterion (Lemma 7.4), f is Riemann integrable. ■

Generalization: Even though continuity guarantees integrability, the converse is not true. A function can be Riemann integrable without being continuous everywhere.

Theorem 52 (Generalization) *Let $f : [a, b] \to \mathbb{R}$ be bounded and have only* ***finitely many points of discontinuity****. Then f is Riemann integrable.*

Sketch of proof. Let $D = \{c_1, c_2, \ldots, c_m\} \subset [a, b]$ be the (finite) set of discontinuities of f. Around each c_i, construct an interval of length less than δ/m such that the total contribution to the upper-lower sum difference over these intervals is less than $\varepsilon/2$. On the complement of these intervals, f is continuous, so we apply the previous theorem to choose a partition on that region giving error less than $\varepsilon/2$. Combining both partitions yields a global partition P such that $U(f, P) - L(f, P) < \varepsilon$. ■

Example 53 (Discontinuous but integrable vs non-integrable) *This example illustrates how the nature and number of discontinuities affect integrability.*

(a) ***Integrable with one discontinuity:*** *Define $f : [-1, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

This function is discontinuous only at a single point $x = 0$, and is zero elsewhere. For any partition that isolates a small interval around 0, say $P_n = \{-1, -\frac{1}{2n}, \frac{1}{2n}, 1\}$, we have:

$$L(f, P_n) = 0, \quad U(f, P_n) = \frac{1}{n} \to 0.$$

Hence,

$$\int_{-1}^{1} f(x)\, dx = 0,$$

and f is integrable even though discontinuous at one point.

(b) ***Not integrable:*** *Define $f : [0, 1] \to \mathbb{R}$ by:*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is known as the Dirichlet function and is discontinuous at ***every point*** *in $[0, 1]$. On every subinterval of any partition:*

$$\inf f = 0, \quad \sup f = 1,$$

so:

$$L(f, P) = 0, \quad U(f, P) = 1 \quad \textit{for all } P.$$

Therefore,

$$U(f, P) - L(f, P) = 1 \not\to 0,$$

and f is not Riemann integrable.

Theorem 54 *Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. In that case:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Remark 55 *If f is integrable on $[a,b]$, we define:*

$$\int_a^b f = -\int_b^a f.$$

Also, for any $c \in [a,b]$, we define:

$$\int_c^c f = 0.$$

Then, for any three points $a,b,c \in I$, where $I \subseteq \mathbb{R}$ is a compact interval and $f : I \to \mathbb{R}$ is integrable, we have:

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

We leave the verification as an exercise.

Theorem 56 (Linearity, Order, and Absolute Value Properties of the Riemann Integral) *Suppose f and g are Riemann integrable on $[a,b]$, and let $k \in \mathbb{R}$. Then:*

1. *The function $f+g$ is integrable, and*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *The function kf is integrable, and*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

3. *If $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

4. *The function $|f|$ is integrable, and*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. We prove parts (1) and (4). Parts (2) and (3) follow from similar arguments and are left as exercises.

(1) Linearity of the integral. Let f and g be integrable on $[a,b]$, and let P be any partition of $[a,b]$ into subintervals $[x_{k-1},x_k]$, $k = 1,\ldots,n$.

Define:

$$m_k^f = \inf_{x\in[x_{k-1},x_k]} f(x), \quad M_k^f = \sup_{x\in[x_{k-1},x_k]} f(x),$$

and similarly for g, and for $f+g$:

$$m_k^{f+g} = \inf_{x\in[x_{k-1},x_k]} (f(x)+g(x)), \quad M_k^{f+g} = \sup_{x\in[x_{k-1},x_k]} (f(x)+g(x)).$$

From basic properties of infima and suprema over sets:

$$m_k^f + m_k^g \leq m_k^{f+g}, \quad M_k^{f+g} \leq M_k^f + M_k^g.$$

Multiplying by the subinterval length $\Delta x_k = x_k - x_{k-1}$, and summing over all k, we obtain:

$$L(f,P) + L(g,P) \le L(f+g,P), \quad U(f+g,P) \le U(f,P) + U(g,P).$$

Let $\varepsilon > 0$. Since f and g are integrable, there exist partitions P_1 and P_2 such that:

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \quad U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$, a common refinement. Then using monotonicity of upper and lower sums under refinement:

$$U(f,P) \le U(f,P_1), \quad L(f,P) \ge L(f,P_1), \quad \text{and similarly for } g.$$

Then:

$$U(f+g,P) \le U(f,P) + U(g,P) \le U(f,P_1) + U(g,P_2) < U(f) + U(g) + \varepsilon,$$
$$L(f+g,P) \ge L(f,P) + L(g,P) \ge L(f,P_1) + L(g,P_2) > L(f) + L(g) - \varepsilon.$$

Thus:

$$U(f+g) \le U(f) + U(g), \quad L(f+g) \ge L(f) + L(g),$$

and since:

$$L(f+g) \le U(f+g),$$

we conclude that:

$$L(f+g) = U(f+g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

So $f+g$ is integrable and its integral is the sum of the integrals.

(4) Integrability of $|f|$ and inequality.

First, note that since f is integrable, it is bounded, say $|f(x)| \le M$ for all $x \in [a,b]$. Let P be a partition of $[a,b]$. Define:

$$m_k^{|f|} = \inf_{x \in [x_{k-1},x_k]} |f(x)|, \quad M_k^{|f|} = \sup_{x \in [x_{k-1},x_k]} |f(x)|.$$

Since $|f(x)|$ is Lipschitz continuous with respect to $f(x)$ (triangle inequality), we have:

$$M_k^{|f|} - m_k^{|f|} \le M_k^f - m_k^f.$$

Summing over all subintervals gives:

$$U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P).$$

Now, since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that:

$$U(f,P) - L(f,P) < \varepsilon \quad \Rightarrow \quad U(|f|,P) - L(|f|,P) < \varepsilon.$$

So $|f|$ is also integrable.

To prove the inequality:

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx,$$

observe that for all $x \in [a,b]$:

$$-|f(x)| \le f(x) \le |f(x)|.$$

Integrating all parts and using the order property (proved in part 3), we get:

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

which implies:

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

■

4.The Fundamental Theorem of Calculus

This central theorem states that the operations of differentiation and integration are, in a sense, inverses of each other.

Theorem 57 (Fundamental Theorem of Calculus)

1. *Let $f : [a, b] \to \mathbb{R}$ be integrable and let $F : [a, b] \to \mathbb{R}$ be differentiable with $F'(x) = f(x)$ for all $x \in [a, b]$. Then*
$$\int_a^b f = F(b) - F(a).$$
2. *Let $g : [a, b] \to \mathbb{R}$ be integrable and define*
$$G(x) := \int_a^x g(t)\,dt, \quad x \in [a, b].$$
Then G is continuous on $[a, b]$. Moreover, if g is continuous at $c \in [a, b]$, then G is differentiable at c, and
$$G'(c) = g(c).$$

In part (1), the function F is called an **antiderivative** of f. In part (2), the function G is called the **indefinite integral** of g.

Remark 58 *Not every derivative is continuous . However, Theorem 57 guarantees that every continuous function is the derivative of some function.*

Proof of Theorem 57. **(1)** Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the Mean Value Theorem, for each interval $[x_{k-1}, x_k]$, there exists $t_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Since $m_k \le f(t_k) \le M_k$, we get

$$L(f, P) \le \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \le U(f, P).$$

The sum $\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$ is a telescoping sum equal to $F(b) - F(a)$, hence

$$\int_a^b f = F(b) - F(a).$$

(2) Suppose $|g(x)| \le M$ on $[a, b]$. For any $x, y \in [a, b]$,

$$|G(x) - G(y)| = \left| \int_a^x g - \int_a^y g \right| = \left| \int_y^x g \right| \le \left| \int_y^x |g| \right| \le M|x - y|.$$

This shows that G is uniformly continuous.

Now suppose g is continuous at $c \in [a, b]$. Then for $x \ne c$:

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x - c} \int_c^x g(t)\,dt.$$

Given $\varepsilon > 0$, by continuity of g at c, there exists $\delta > 0$ such that $|g(t) - g(c)| < \varepsilon$ whenever $|t - c| < \delta$. Then for $0 < |x - c| < \delta$:

$$\left| \frac{G(x) - G(c)}{x - c} - g(c) \right| = \left| \frac{1}{x - c} \int_c^x (g(t) - g(c))\,dt \right| \le \varepsilon.$$

Hence $G'(c) = g(c)$. ■

Remark 59 *Computing integrals directly from the definition is usually not feasible in practice. The power of the Fundamental Theorem lies in allowing us to compute definite integrals using antiderivatives.*

Theorem 60 (Mean Value Theorem for Integrals) *If $g : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in (a, b)$ such that*

$$\int_a^b g = (b - a)g(c).$$

Proof. Apply the Mean Value Theorem to the function $x \mapsto \int_a^x g$, which by the Fundamental Theorem of Calculus is an antiderivative of g. ■

Improper Integrals

In this section, we study improper integrals, which arise in two main situations:

- One of the integration limits is infinite,
- The function becomes unbounded (e.g., has a vertical asymptote) at a boundary point.

We will consider these two cases in detail.

Case 1: Integration over an Infinite Interval

Definition 61 *Let $f : [a, \infty) \to \mathbb{R}$ be a function that is Riemann integrable over every finite interval $[a, R]$, for $a < R < \infty$. If the limit*

$$\lim_{R\to\infty} \int_a^R f(x)\, dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^\infty f(x)\, dx := \lim_{R\to\infty} \int_a^R f(x)\, dx.$$

Similarly, for a function $f : (-\infty, a] \to \mathbb{R}$, we define

$$\int_{-\infty}^a f(x)\, dx := \lim_{R\to-\infty} \int_R^a f(x)\, dx,$$

provided the limit exists.

Example

Consider the integral

$$\int_1^\infty \frac{1}{x^s}\, dx.$$

We compute:

$$\int_1^R \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}\left(1 - \frac{1}{R^{s-1}}\right), & s \neq 1, \\ \log R, & s = 1. \end{cases}$$

Taking the limit as $R \to \infty$, we get:

$$\int_1^\infty \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}, & \text{if } s > 1, \\ \text{diverges}, & \text{if } s \leq 1. \end{cases}$$

Case 2: The Function is Unbounded at an Endpoint

Definition 62 *Let $f : (a,b] \to \mathbb{R}$ be a function that is Riemann integrable over every interval $[a+\varepsilon, b]$, for $0 < \varepsilon < b-a$. If the limit*

$$\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx.$$

Example

Let us evaluate

$$\int_0^1 \frac{1}{x^s}\,dx.$$

For $s \neq 1$, we compute:

$$\int_\varepsilon^1 \frac{1}{x^s}\,dx = \frac{1}{1-s}\left(1 - \varepsilon^{1-s}\right).$$

Now take the limit as $\varepsilon \to 0^+$:

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1-s} = \begin{cases} 0, & s < 1, \\ \infty, & s > 1. \end{cases}$$

Hence,

$$\int_0^1 \frac{1}{x^s}\,dx = \begin{cases} \dfrac{1}{1-s}, & \text{if } s < 1, \\ \text{diverges}, & \text{if } s \geq 1. \end{cases}$$

We now consider the general case of improper integrals over open intervals.

Definition 63 *Let $f : (a,b) \to \mathbb{R}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, be a function that is Riemann integrable over every compact subinterval $[\alpha, \beta] \subset (a,b)$. Let $c \in (a,b)$ be arbitrary. If both of the improper integrals*

$$\int_a^c f(x)\,dx := \lim_{\alpha \searrow a} \int_\alpha^c f(x)\,dx, \quad \int_c^b f(x)\,dx := \lim_{\beta \nearrow b} \int_c^\beta f(x)\,dx$$

converge, then the integral over the full interval is called convergent, and we define:

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note that this definition is independent of the choice of the intermediate point $c \in (a,b)$.

Examples

Example 1

According to previous examples, the integral

$$\int_0^\infty \frac{1}{x^s}\,dx$$

diverges for all $s \in \mathbb{R}$.

Example 2

The integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

converges. We compute:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{\varepsilon \searrow 0} \int_{-1+\varepsilon}^{0} \frac{1}{\sqrt{1-x^2}}\, dx + \lim_{\varepsilon \searrow 0} \int_{0}^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\, dx \\
&= -\lim_{\varepsilon \searrow 0} \sin^{-1}(-1+\varepsilon) + \lim_{\varepsilon \searrow 0} \sin^{-1}(1-\varepsilon) \\
&= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi.
\end{aligned}$$

Example 3

The integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

also converges:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx &= \lim_{R \to \infty} \int_{-R}^{0} \frac{1}{1+x^2}\, dx + \lim_{R \to \infty} \int_{0}^{R} \frac{1}{1+x^2}\, dx \\
&= -\lim_{R \to \infty} \tan^{-1}(-R) + \lim_{R \to \infty} \tan^{-1}(R) \\
&= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi.
\end{aligned}$$