Chapter 1 Fundamentals

§ Real numbers : $\mathbb{R}$

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} \ ; \ a \in \mathbb{Z} \ ; \ b \in \mathbb{Z}^* \right\}$$

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

Properties of real numbers

① $a+b = b+a$
$a \cdot b = b \cdot a$

② $a+(b+c) = (a+b)+c$
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

③ $a \cdot (b+c) = a \cdot b + a \cdot c$

exple. $2(x+3) = 2x+6$
$(a+b)(x+y) = ax+ay+bx+by$

④ $-(a+b) = -a-b$
$-(a-b) = -a+b$

exple $-(x+2) = -x-2$
$-(x+y-z) = -x-y+z$

Properties of fractions

① $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

expl: $\frac{2}{3} + \frac{3}{5} = \frac{10+9}{15} = \frac{19}{15}$

$\frac{4}{7} - \frac{2}{3} = \frac{12-14}{21} = \frac{-2}{21}$

② $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

③ $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

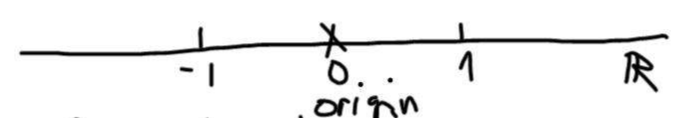
④ $\frac{a/b}{c/d} = \frac{ad}{bc}$

⑤ $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

exple $\frac{5}{36} + \frac{7}{120} = \frac{50+21}{8 \times 9 \times 5}$ $\quad 120 = 2^3 \cdot 5 \cdot 3$, $\ 36 = 2^2 \cdot 3^2$

$= \frac{71}{360}$

- The real line



- Sets & Intervals

A set is a collection of objects and these objects are called the element of a set.

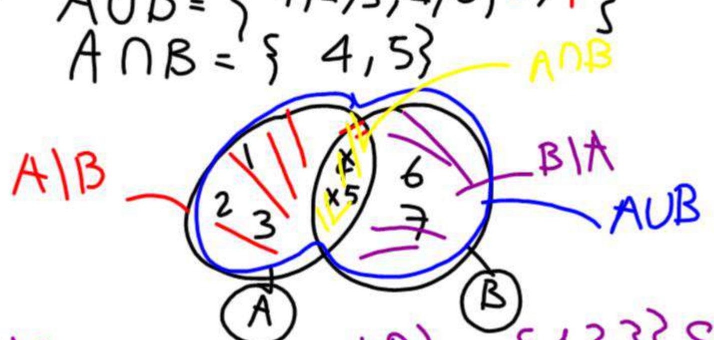
If S is a set, $a \in S$ means that a is an element of S

let $A = \{1, 2, 3, 4, 5\}$
$B = \{4, 5, 6, 7\}$

We define the union of 2 sets A & B as $A \cup B = \{x \ / \ x \in A \text{ or } x \in B\}$

Also the intersection of 2 sets A & B as $A \cap B = \{x \ / \ x \in A \text{ and } x \in B\}$

exple $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$A \cap B = \{4, 5\}$



$A \setminus B = \{x \in A \ / \ x \notin B\} = \{1, 2, 3\} \subseteq A$

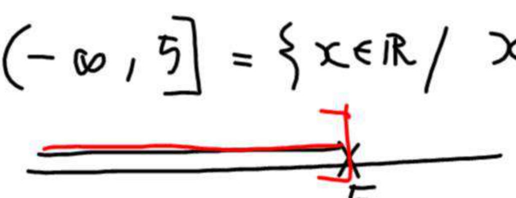
⚠ $A \cup B = (A \setminus B) \sqcup (A \cap B) \sqcup (B \setminus A)$
partition

$\#(A \cup B) = \#A + \#B - \#(A \cap B)$
$7 = 5 + 4 - 2$

Def (Interval)
If $a < b$ then the open Interval from a to b consists of all numbers between a & b
We denoted it $(a, b) = \{x \in \mathbb{R} / a < x < b\}$

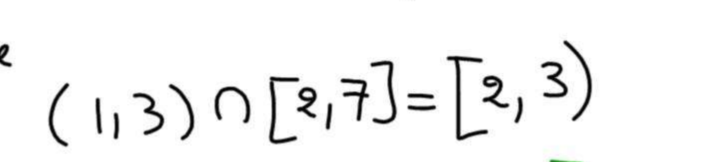
The closed interval (segment)
$[a, b] = \{x \in \mathbb{R} \ / \ a \leq x \leq b\}$

exple. $(-\infty, 5] = \{x \in \mathbb{R} \ / \ x \leq 5\}$



- $\mathbb{R} = (-\infty, +\infty)$

exple
- $(1,3) \cap [2,7] = [2, 3)$



- $(1,3) \cup [2,7] = (1, 7]$

* Absolute value

Def $|x| = \begin{cases} x & \text{if } x \geq 0 \\ (-x) & \text{if } x < 0 \end{cases}$

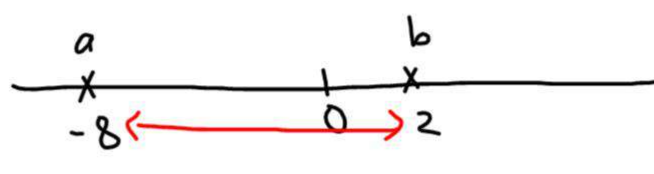
exple $|-5| = -(-5) = 5$
$|0| = 0$
$|2| = 2$
$|3-\pi| = -(3-\pi) = \pi - 3 > 0$

Properties:
- $|x| \geq 0$ for all $x \in \mathbb{R}$
- $|-x| = |x|$
- $|xy| = |x| \cdot |y|$
- $|x+y| \leq |x| + |y|$ (Triangle inequality)

⚠ If a & b are real numbers, then the distance between a & b is

$$d(a, b) = |a-b|$$

exple $a(-8)$ & $b(2)$
then $d(a, b) = |-8-2| = |-10| = 10$



Exponents & Radicals

Def: If a is any real number and n an integer then the n^{th} power of a

$$a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}$$

exple
$$\left(\frac{1}{2}\right)^5 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{32}$$
$$(-3)^4 = (-3)(-3)(-3)(-3) = 81$$
$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$
$$5^0 = 1$$

Laws of exponents

- $a^m \cdot a^n = a^{m+n}$ exple $2^5 \cdot 2^3 = 2^8$
- $(a^m)^n = a^{m\cdot n}$ exple $(2^3)^4 = 2^{12}$
- $\frac{a^m}{a^n} = a^{m-n}$ exple $\frac{3^5}{3^7} = 3^{5-7} = 3^{-2} = \frac{1}{9}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

exple
- $x^4 \cdot x^7 = x^{11}$
- $y^4 \cdot y^{-7} = y^{-3}$
- $\frac{c^9}{c^5} = c^{9-5} = c^4$
- $(b^4)^5 = b^{20}$
- $(3x)^3 = 3^3 \cdot x^3 = 27x^3$
- $\left(\frac{x}{2}\right)^4 = \left(\frac{1}{2}\right)^4 x^4 = \frac{1}{16}x^4$
- $(2a^3b^2)(3ab^4)^3 = 2a^3b^2 \cdot 27 \cdot a^3 \cdot b^{12} = 54a^6b^{14}$
- $\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4 = \frac{x^3}{y^3}\cdot\frac{y^8x^4}{z^4} = \frac{x^7y^8}{y^3z^4} = \frac{x^7y^5}{z^4}$

- **Radicals**:

The symbol $\sqrt{\ }$ means the positive square root

$$\boxed{\sqrt{a} = b \Leftrightarrow a = b^2}$$
$$\sqrt{9} = 3 \Leftrightarrow 9 = 3^2$$

⚠ if n is a positive integer then

$$\boxed{\sqrt[n]{a} = b \Leftrightarrow a = b^n}$$

⚠ $\sqrt{a^2} = |a|$; $\sqrt{(-3)^2} = |-3| = 3$

$$\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$$

$\sqrt[n]{a^n} = |a|$ if n even (زوجي)

$\sqrt[n]{a^n} = a$ if n odd (فردي)

- $\sqrt[n]{ab} = \sqrt[n]{a}\cdot\sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = a^{\frac{1}{mn}}$

exple:
- $\sqrt[3]{x^4} = x^{4/3} = x^{1+\frac{1}{3}} = x^1 \cdot x^{1/3} = x\sqrt[3]{x}$
- $\sqrt[4]{81x^8y^4} = (81x^8y^4)^{1/4} = (3^4x^8y^4)^{1/4} = 3x^2y$

⚠ $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ if n even

- $4^{1/2} = \sqrt{4} = 2$
- $(125)^{-1/3} = \sqrt[3]{\frac{1}{125}} = \sqrt[3]{\left(\frac{1}{5}\right)^3} = \frac{1}{5}$
- $8^{2/3} = (2^3)^{2/3} = 2^2 = 4$
- $a^{1/3} \cdot a^{7/3} = a^{8/3}$
- $\frac{a^{2/5} \cdot a^{7/5}}{a^{3/5}} = \frac{a^{9/5}}{a^{3/5}} = a^{9/5 - 3/5} = a^{6/5}$
- $\frac{1}{\sqrt[3]{x^4}} = \frac{1}{x^{4/3}} = x^{-4/3}$
- $2\sqrt{x}\ 3\sqrt[3]{x} = 6x^{1/2} \cdot x^{1/3} = 6x^{\frac{1}{2}+\frac{1}{3}} = 6x^{5/6}$
- $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3}$
- $\frac{1}{\sqrt[3]{5}} = \frac{1}{5^{1/3}} \div \frac{5^{2/3}}{5^{1/3}\cdot 5^{2/3}} = \frac{5^{2/3}}{5} = \frac{\sqrt[3]{25}}{5}$

- **Special Product Formulas**

$-1^2 = -1$

$(-1)^2 = 1$

① $(a-b)(a+b) = a^2 - b^2$

$(x-1)(x+1) = x^2 - 1^2 = x^2 - 1$

② $(a+b)^2 = a^2 + 2ab + b^2$

$$\boxed{a^2 + 2ab = (a+b)^2 - b^2}$$

complete square formula

$$(a-b)^2 = a^2 - 2ab + b^2$$

③ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

④ $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

exple
- $(x^2+y^3)^2 = x^4 + 2x^2y^3 + y^6$
- $(3x-5)^2 = 9x^2 - 30x + 25$
- $(\boxed{x^2} - \boxed{2})(\boxed{x^2} + \boxed{2}) = x^4 - 4$
- $(x^2-2)^3 = x^6 - 3x^4 \cdot 2 + 3x^2 \cdot 4 - 8 = x^6 - 6x^4 + 12x^2 - 8$

$a = x^2$

$b = 2$

exple.

- $(2x-\sqrt{y})(2x+\sqrt{y}) = 4x^2 - y$
- $(\boxed{x+y}-1)(x+y+1) = (x+y)^2 - 1$

$$= x^2+2xy+y^2-1$$

exple: Factor each expression:

- $3x^2-6x = 3x(x-2)$
- $(2x+4)(x-3)-5(x-3) = (x-3)(2x+4-5)$

$$= (x-3)(2x-1)$$

- $8x^4y^2+6x^3y^3-2xy^4 =$

$$2xy^2\left[4x^3+3x^2y-y^2\right]$$

- $4x^2-25 = \boxed{(2x)}^2 - 5^2$

$$= (\square - \Delta)(\square + \Delta)$$

$$= (2x-5)(2x+5)$$

- $\boxed{(x+y)}^2 - 3^2 = (x+y-3)(x+y+3)$

- $2x^4-8x^2 = 2x^2(\underbrace{x^2-4})$

$$= 2x^2(x-2)(x+2)$$

- $x^5y^2-xy^6 = xy^2(x^4-y^4)$

$$= xy^2(\boxed{(x^2)}^2-(y^2)^2)$$

$$= xy^2(x^2-y^2)(x^2+y^2)$$

$$= xy^2(x-y)(x+y)(x^2+y^2)$$

exercise: Factor by grouping terms

(a) x^3+x^2+4x+4

(b) $x^3-2x^2-9x+18$

(e) $2x = 1-\sqrt{2-x}$ (*) $\quad \sqrt{2-x} = \underbrace{1-2x}_{\geq 0}$

The domain of eq(*) is

$D_{eq} = \{x \in \mathbb{R} \ / \ 2-x \geq 0 \text{ and } 1-2x \geq 0\}$

$= \{x \in \mathbb{R} \ / \ x \leq 2 \text{ and } x \leq 1/2\}$

for $x \leq 2$; $\quad = (-\infty, \frac{1}{2}]$

$2x-1 = -\sqrt{2-x}$

$(2x-1)^2 = 2-x$

$4x^2-4x+1 = 2-x$

$4x^2-3x-1=0$

$(x-1)(4x+1)=0$

$a=4$; $b=-3$; $c=-1$

Discriminant $D = b^2-4ac = 25 = 5^2$

$x_1 = \frac{3-5}{8} = -1/4$; $x_2 = \frac{3+5}{8} = 1 > 1/2$

$S = \{-1/4\}$

(f) $x^4 - 8x^2 + 8 = 0$ (*)

put $x^2 = u$

(*) becomes $u^2-8u+8=0$

$a=1$; $b=-8$; $c=8$

Discriminant $D = b^2-4ac = 64-32 = 32 > 0$
$= (2\sqrt{8})^2$
$= (4\sqrt{2})^2$

$u_1 = \frac{8-4\sqrt{2}}{2} = 4-2\sqrt{2} > 0 = x^2$

$u_2 = \frac{8+4\sqrt{2}}{2} = 4+2\sqrt{2} > 0 = x^2$

$S = \{\pm\sqrt{(4-2\sqrt{2})} \ ; \ \pm\sqrt{4+2\sqrt{2}}\}$

- **Complex numbers** (*) $x^2 = -1$ has no solution in $\mathbb{R}$

So $\mathbb{R} \subset \mathbb{C}$

(*) has solution in $\mathbb{C}$ denoted $x = \pm i$

Def: A complex number is an expression of the form $z = a+ib$ where $a, b \in \mathbb{R}$ and $\boxed{i^2=-1}$

- The real part of z denoted $\operatorname{Re} z = a$
  The imaginary part of z denoted $\operatorname{Im} z = b$

⚠ $z_1 = a_1+ib_1$; $z_2 = a_2+ib_2$

$(z_1 = z_2) \Leftrightarrow \begin{cases} a_1 = a_2 \\ \text{and} \\ b_1 = b_2 \end{cases}$

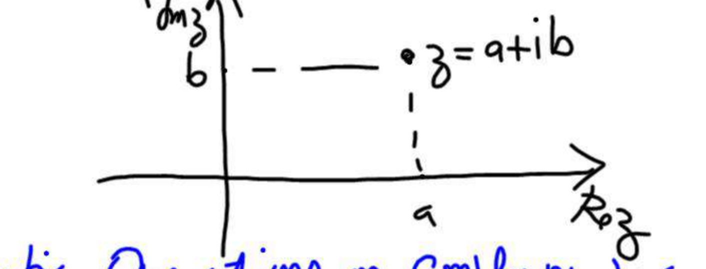
expl

$z_1 = 3+4i$ then $\operatorname{Re} z_1 = 3$ & $\operatorname{Im} z_1 = 4$

$z_2 = 2i$ then $\operatorname{Re} z_2 = 0$ & $\operatorname{Im} z_2 = 2$

$z_3 = -3$ then $\operatorname{Re} z_3 = -3$ & $\operatorname{Im} z_3 = 0$

⚠



- Arithmetic Operations on Complex numbers.
  - Addition
    let $z_1 = a_1+ib_1$ & $z_2 = a_2+ib_2$
    then $z_1+z_2 = (a_1+a_2)+i(b_1+b_2)$
  - Substraction:
    $z_1-z_2 = (a_1-a_2)+i(b_1-b_2)$
  - Multiplication:
    $z_1 \cdot z_2 = (a_1+ib_1)(a_2+ib_2)$
    $= a_1a_2 + ia_1b_2 + ib_1a_2 + \overset{=-1}{i^2} b_1b_2$
    $= (a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)$

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expl: Express the following in the form $a+ib$

(a) $(3+5i)+(4-2i) = 7+3i$

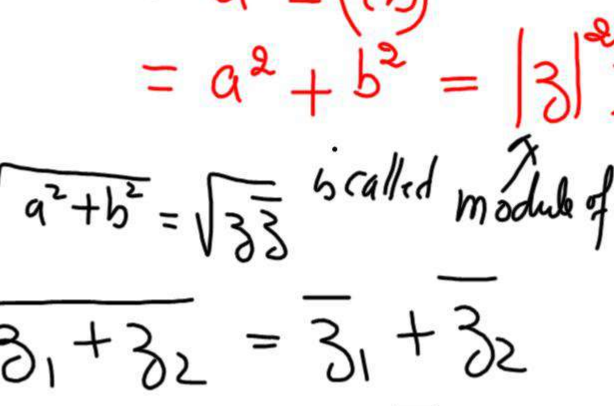
(b) $(3+5i)-(4-2i) = -1+7i$

(c) $(3+5i)(4-2i) = 12-6i+20i+10$
$= 22-14i$

(d) $i^{23} = (i)^{22+1} = i^{22}\, i$
$= (i^2)^{11}\, i = (-1)^{11} \cdot i = -i$

Def (Conjugate)
المرافق

Let $z = a+ib \in \mathbb{C}$, we define the conjugate of z as $\bar{z} = a-ib$



⚠ $z\bar{z} = (a+ib)(a-ib)$
$= a^2-(ib)^2$
$= a^2+b^2 = |z|^2 \geq 0$

- $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$ is called module of z

⚠
- $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\bar{\bar{z}} = z$
- $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{z^2} = (\bar{z})^2$

⚠ to simplify $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}$

expl Express the following in the form $a+ib$

(a) $\frac{3+5i}{1-2i} = \frac{(3+5i)(1+2i)}{(1-2i)(1+2i)}$

$= \frac{3+6i+5i-10}{1^2+2^2}$

$= -7/5 + i\,\frac{11}{5}$

(b) $\frac{7+3i}{i} = \frac{(7+3i)(-i)}{i\,(-i)} = \frac{-7i+3}{1}$

$= 3-7i$

⚠ Let $r \geq 0$; $\sqrt{-r} = \pm i\sqrt{r}$

exple $\sqrt{-9} = \pm i\sqrt{9} = \pm 3i$

$\sqrt{-16} = \pm i\sqrt{16} = \pm 4i$

⚠ $ax^2 + bx + c = 0$ with

$a \neq 0$; $a, b, c \in \mathbb{R}$

and we suppose $D = b^2 - 4ac < 0$

Then $X_1 = \dfrac{-b - i\sqrt{-D}}{2a}$; $X_2 = \overline{X_1} = \dfrac{-b + i\sqrt{-D}}{2a}$

exple: $x^2 + 4x + 5 = 0$

$a = 1$; $b = 4$; $c = 5$

Discriminant $D = b^2 - 4ac = -4$

$\sqrt{D} = \pm 2i \quad = (2i)^2$

$x_1 = \dfrac{-b - i\sqrt{-D}}{2a} = \dfrac{-4 - 2i}{2} = -2 - i$

$x_2 = \dfrac{-b + i\sqrt{-D}}{2a} = -2 + i$

exple Solve in $\mathbb{C}$,

(a) $x^2 + 9 = 0$

(b) $4x^2 - 24x + 37 = 0$

Ans : (a) $x^2 + 9 = 0 \Rightarrow x^2 = -9$

$x^2 = (i^2)9$

$= (\pm 3i)^2$

$\boxed{x = \pm 3i}$

[$\square^2 = -r$ with $r \geq 0$.

So $\square = \pm i\sqrt{r}$

(b) $4x^2 - 24x + 37 = 0$

$a = 4$; $b = -24$; $c = 37$

discriminant $D = b^2 - 4ac$

$= (-24)^2 - 4 \cdot 4 \cdot 37$

$= 576 - 592$

$= -16 = (4i)^2$

$x_1 = \dfrac{24 - 4i}{8} = 3 - \frac{1}{2}i = -\dfrac{b - i\sqrt{-D}}{2a}$

$x_2 = \dfrac{24 + 4i}{8} = 3 + \frac{1}{2}i = \dfrac{-b + i\sqrt{-D}}{2a}$

⚠ if z is a solution for $P(X) = 0$

with P is a polynomial function

where all coefficient are real numbers

then $\overline{z}$ is also a solution for $P(X) = 0$.

Exponential and logarithmic Functions.

§ Exponential function

Def: The exponential function with base a is defined for all real numbers x by $f(x) = a^x$ with $a>0$, $a \neq 1$

exple $f(x) = 2^x$; $g(x) = 10^x$

$h(x) = e^x$ (natural exponential function)

$k(x) = \left(\frac{1}{5}\right)^x$

Graph of exponential functions

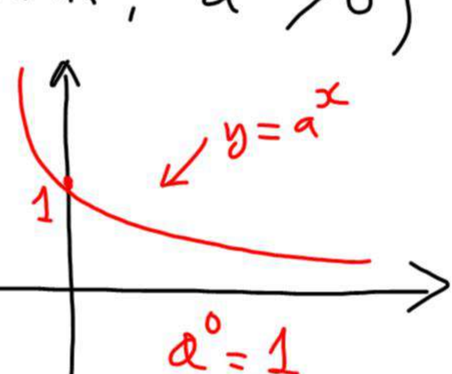
$f(x) = a^x$ with $a>0$; $a\neq 1$

domain of f is $\mathbb{R}$

Range of f is $(0,+\infty)$

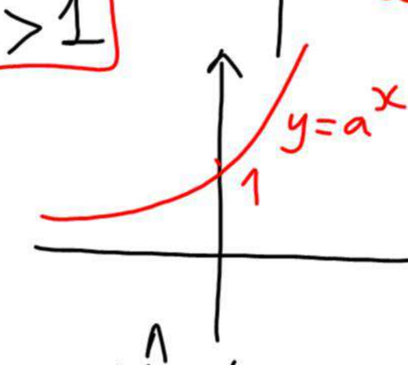
(For $x\in\mathbb{R}$; $a^x>0$)

• if $\boxed{0<a<1}$

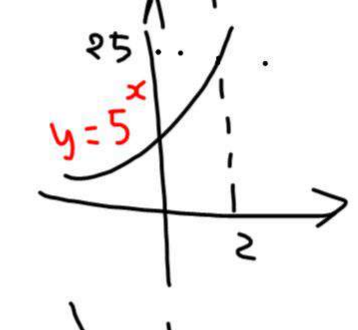


$a^0 = 1$

• if $\boxed{a>1}$



exple ①

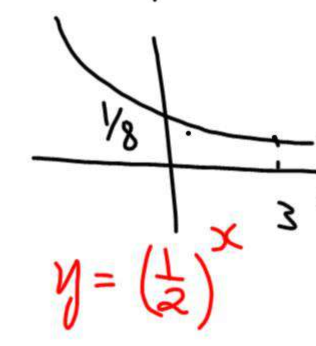


$f(x) = a^x$; $a>1$

$f(2) = 25$

So $a^2 = 25 \Rightarrow a=5$

②



$y = \left(\frac{1}{2}\right)^x$

$f(x) = a^x$; $0<a<1$

$f(3) = \frac{1}{8}$

$\Leftrightarrow a^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3$

So $a = \frac{1}{2}$

Properties:

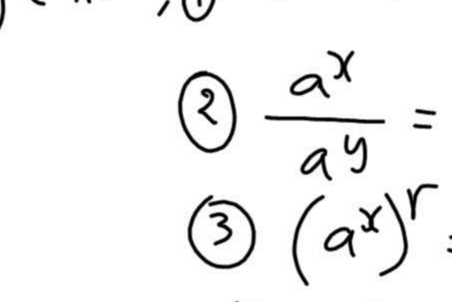
For $x, y \in \mathbb{R}$; ① $a^x \cdot a^y = a^{x+y}$

② $\frac{a^x}{a^y} = a^{x-y}$

③ $(a^x)^r = a^{xr}$ with $r\in\mathbb{R}$

exple • $f(x) = e^{-x} = \frac{1}{e^x} = \left(\frac{1}{e}\right)^x$

$(e^{-x}) = (e^x)^{-1} = \frac{1}{e^x}$



$e \approx 2,761$

• $\left(e^{x/2}\right)^2 = e^{x/2} \cdot e^{x/2} = e^x$

Def: (logarithm functions)

Let a be a positive number with $a\neq 1$

The logarithm function with base a, denoted by $\log_a$ is defined by

$y = \log_a(x) \Leftrightarrow x = a^y > 0$

⚠ Domain $\log_a = (0,\infty)$

Range $\log_a = \mathbb{R}$

exple:

• $\log_{10}(\underbrace{1000}_{x}) = \underset{y=?}{3} \Leftrightarrow \underbrace{1000}_{x} = 10^3$

• $\log_2(32) = ?\,5 \Leftrightarrow 32 = 2^{\boxed{5}}$

• $\log_{10}(0,1) = -1 \Leftrightarrow 0,1 = 10^{-1}$

• $\log_{16}(4) = \frac{1}{2} \Leftrightarrow 4 = 16^{1/2} = \sqrt{16}$

⚠ $\log_a(a^x) = x$, $x\in\mathbb{R}$

$a^{\log_a(x)} = x$; $x>0$

Properties of logarithms

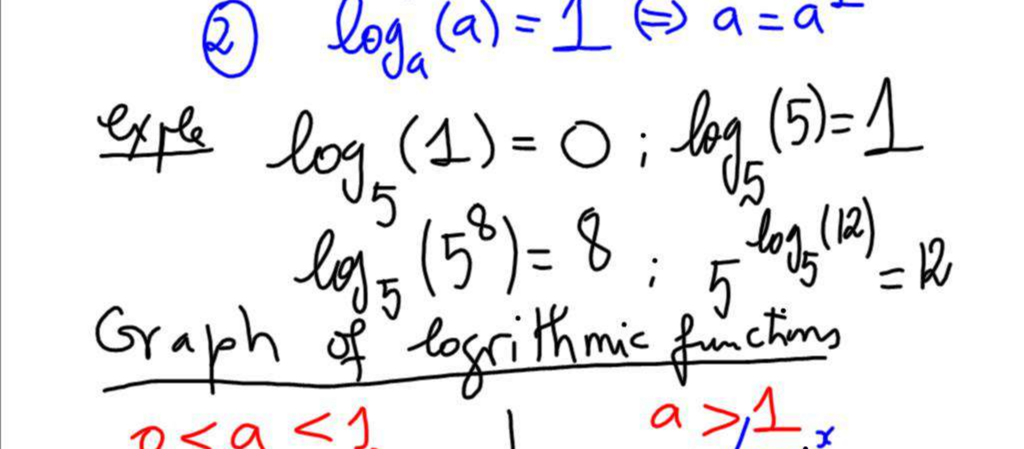
① $\log_a(1) = 0 \Leftrightarrow a^0 = 1$

② $\log_a(a) = 1 \Leftrightarrow a = a^1$

exple $\log_5(1) = 0$; $\log_5(5) = 1$

$\log_5(5^8) = 8$; $5^{\log_5(12)} = 12$

Graph of logrithmic functions



⚠ - Common logrithm

The logarith with base 10 is called the common logarithm is denoted $\log(x)$

$\log 10 = 1$; $\log(100) = 2$

- Natural logarithm

The logarithm with base e is called the natural logarithm denoted $\ln(x)$

$\log_e(x) = \ln(x)$ for $x>0$

$y = \ln x \Leftrightarrow x = e^y$

• $\ln 1 = 0$; $\ln(e) = 1$

• $\ln(e^x) = x$ for $x\in\mathbb{R}$

• $e^{\ln x} = x$ for $x>0$

exple $\ln(e^8) = 8$; $\ln\left(\frac{1}{e^2}\right) = \ln(e^{-2}) = -2$

exple Find the domain of $f(x) = \ln(4-x^2)$

Ans: The domain of f is $D_f = \{x\in\mathbb{R} / 4-x^2>0\}$

$= (-2,2)$

x	$-\infty$		-2		2		$+\infty$
Mark $(4-x^2)$		$-$	0	$+$	0	$-$	

Properties Let $x, y > 0$. Then

① $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

② $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

③ $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$

④ $\log_a(x^r) = r\log_a(x); \forall r \in \mathbb{R}$

• Relation between natural logarithm & general logarithm function.

$$\boxed{\log_a(x) = \frac{\ln x}{\ln a}; \ x > 0}$$

So $\log_b(x) = \frac{\ln x}{\ln b} = \boxed{\frac{\ln a}{\ln b}} \frac{\ln x}{\ln a}$

$$\boxed{\log_b(x) = \log_b(a)\ \log_a(x)}$$

exple Solve

① $5^x = 125$

② $5^{2x} = 5^{x+1}$

③ $3^{x+2} = 7$

④ $8e^{2x} = 20$

⑤ $e^{3-2x} = 4$

⑥ $e^{2x} - e^x - 6 = 0$

⑦ $3xe^x + x^2e^x = 0$

⑧ $\ln x = 8$

⑨ $\log(x^2+1) = \log(x-2) + \log(x+3)$

⑩ $\log_2(25-x) = 3$

⑪ $\log(x+2) + \log(x-1) = 1$

Ans

① $5^x = 125$ then $\log_5(5^x) = \log_5(125)$

$x = \log_5(5^3) = 3$

② $5^{2x} = 5^{x+1}$

$\log_5(5^{2x}) = \log_5(5^{x+1})$

$2x = x+1$

$2x - x = 1 \Leftrightarrow \boxed{x = 1}$

③ $3^{x+2} = 7$

so $\log_3(3^{x+2}) = \log_3(7)$

$x + 2 = \log_3(7) = \frac{\ln 7}{\ln 3}$

$\boxed{x = \frac{\ln 7}{\ln 3} - 2}$

④ $8e^{2x} = 20$

$e^{2x} = \frac{20}{8} = \frac{5}{2}$

$\ln(e^{2x}) = \ln\left(\frac{5}{2}\right)$

$2x = \ln\left(\frac{5}{2}\right)$

$x = \frac{1}{2}\ln\left(\frac{5}{2}\right) = \frac{1}{2}[\ln 5 - \ln 2]$

⑤ $e^{3-2x} = 4$

So $\ln(e^{3-2x}) = \ln(4)$

$3 - 2x = \ln 4$

$3 - \ln 4 = 2x \Leftrightarrow x = \frac{1}{2}(3 - \ln 4)$

⑥ $e^{2x} - e^x - 6 = 0$

(*) $(e^x)^2 - (e^x) - 6 = 0$

put $u = e^x > 0$; (*) becomes

$u^2 - u - 6 = 0$

$a = 1; b = -1; c = -6$

Discriminant $D = b^2 - 4ac = 1 + 24 = 25 = 5^2 > 0$

$u_1 = \frac{-b-\sqrt{D}}{2a} = \frac{1-5}{2} = -2 = e^x$ ↯

$u_2 = \frac{-b+\sqrt{D}}{2a} = \frac{1+5}{2} = 3 = e^x$

so $\boxed{x = \ln 3}$

⑦ $3xe^x + x^2e^x = 0$

$xe^x[3+x] = 0$ so $x = 0$ or $x = -3$

⑧ $\log(x^2+1) = \log(x-2) + \log(x+3)$

Domain of equation $D_{eq} = \{x \in \mathbb{R} / \ x^2+1>0,\ x-2>0,\ x+3>0\}$

$= \{x \in \mathbb{R};\ x > 2\}$

$= (2, +\infty)$

let $x > 2$,

$\log(x^2+1) = \log((x-2)(x+3))$

$x^2 + 1 = (x-2)(x+3)$

$\cancel{x^2} + 1 = \cancel{x^2} + x - 6$

$\boxed{x = 7 > 2}$

⑨ $\ln x = 8$; $x > 0$

$e^{\ln x} = e^8$

$\boxed{x = e^8}$

⑩ $\log_2(25-x) = 3$

$D_{eq} = (-\infty, 25)$

let $x < 25$:

$2^{\log_2(25-x)} = 2^3$

$25 - x = 8$

$\boxed{x = 17}$

⑪ $\log(x+2) + \log(x-1) = 1$

The domain of equation is

$D_{eq} = \{x \in \mathbb{R} / \ x+2>0 \text{ and } x-1>0\}$

$= (-2, +\infty) \cap (1, +\infty) = (1, +\infty)$

let $x > 1$

$\log(x+2) + \log(x-1) = \log((x+2)(x-1)) = 1$

$10^{\log(x^2+x-2)} = 10$

$x^2 + x - 2 = 10$

$x^2 + x - 12 = 0$

$a = 1; b = 1; c = -12$

discriminant $D = b^2 - 4ac = 1 + 48 = 7^2 > 0$

$x_1 = \frac{-b-\sqrt{D}}{2a} = \frac{-1-7}{2} = -4 < 1$ ↯

$x_2 = \frac{-b+\sqrt{D}}{2a} = \frac{-1+7}{2} = 3 > 1$

The solution is $x = 3$.

5 Differentiation of a^x & $\log_a(x)$:

Def: Let f be a function defined, continuous on an Interval $[a,b]$ and let $x_0 \in [a,b]$. We say that f is differentiable at $x=x_0$ if $\lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ exists $= \left.\dfrac{df}{dx}\right|_{x=x_0}$

⚠ When we put $h = x - x_0$;

$$\left.\frac{df}{dx}\right|_{x=x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

expl ① $f(x) = 5x+7$. Use the definition to find $f'(1)$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{(5(1+h)+7)-12}{h}$$
$$= \lim_{h\to 0} \frac{5h}{h} = 5$$

⚠ $f'(x) = 5$

② $f(x) = 3x^2+1$, using the definition of derivative to find $f'(x)$.

Ans $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2+1-(3x^2+1)}{h}$$
$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)+\cancel{1}-\cancel{3x^2}-\cancel{1}}{h}$$
$$= \lim_{h\to 0} \frac{6xh+3h^2}{h}$$
$$= \lim_{h\to 0} (6x+3h) = 6x$$

⚠ If $f(x) = x^r$ then $f'(x) = rx^{r-1}$

- $\dfrac{d}{dx}(\text{cte}) = 0$; $\dfrac{d}{dx}x = 1$; $\dfrac{d}{dx}(x^2) = 2x$
- $\dfrac{d}{dx}(x^5) = 5x^4$
- $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$
- $\dfrac{d}{dx}(x^{1/7}) = \dfrac{1}{7}x^{\frac{1}{7}-1} = \dfrac{1}{7}x^{-6/7}$

Proposition Let f & g be differentiable on the same interval: Then

① $\dfrac{d}{dx}(f(x)+g(x)) = \dfrac{df}{dx}(x) + \dfrac{dg}{dx}(x) = f'(x)+g'(x)$

② $\dfrac{d}{dx}(c\,f(x)) = c\dfrac{d}{dx}f(x) = c\,f'(x)$ with c is a constant.

expl $f(x) = 2x^5 - 3\sqrt{x} + 4/x^3$

find $f'(x) = ?$

Ans: $f(x) = 2x^5 - 3x^{1/2} + 4x^{-3}$

Then $f'(x) = 2(5x^4) - 3(\frac{1}{2}x^{-1/2}) + 4(-3x^{-4})$

$$= 10x^4 - \frac{3}{2}\frac{1}{\sqrt{x}} - 12\frac{1}{x^4}$$

Prop Let f & g be differentiable on the same interval I then $(f\cdot g)$ is also differentiable on I. Moreover

$$(f\cdot g)'(x) = f'(x)g(x) + f(x)\cdot g'(x)$$

expl Find f' if $f(x) = (x^2+3x-5)(\sqrt{x+1}-7)$

Ans: $f'(x) = \dfrac{d}{dx}(x^2+3x-5)\cdot(\sqrt{x+1}-7) + (x^2+3x-5)\dfrac{d}{dx}(\sqrt{x+1}-7)$

So $f'(x) = (2x+3)(\sqrt{x+1}-7) + (x^2+3x-5)\,\frac{1}{2}(x+1)^{-1/2}$

Prop Let f & g be differentiable on the same Interval I and we suppose that $g(x) \neq 0$ when $x \in I$. Then $\left(\dfrac{f}{g}\right)$ is also differentiable on I. Moreover,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

expl: let $f(x) = \dfrac{2x-3}{x^2+1}$. find $f'(x)$.

Ans: $f'(x) = \dfrac{\frac{d}{dx}(2x-3)\cdot(x^2+1) - (2x-3)\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

$$f'(x) = \frac{2(x^2+1)-(2x-3)(2x)}{(x^2+1)^2}$$
$$= \frac{2x^2+2-4x^2+6x}{(x^2+1)^2}$$
$$= \frac{-2x^2+6x+2}{(x^2+1)^2}$$

⚠ $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = -\dfrac{f'(x)}{(f(x))^2}$

expl.
- $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$
- $\dfrac{d}{dx}\left(\dfrac{1}{5x^2-3x+2}\right) = -\dfrac{\frac{d}{dx}(5x^2-3x+2)}{(5x^2-3x+2)^2} = \dfrac{-[10x-3]}{(5x^2-3x+2)^2}$

Composition of 2 functions

$$\begin{tikzcd} \bigcirc \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \bigcirc \arrow[r, "f"] & \bigcirc \end{tikzcd}$$

$$(f\circ g)(x) = f(g(x))$$

expl If $f(x) = \sqrt{x}$ and $g(x) = 3x^2 - x + 1$

find $f\circ g$ & $g\circ f$

- exponential function: $y = a^x = f(x)$

$$f'(x) = \underbrace{f'(0)}_{\ln a}\, a^x \; ; \quad \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \underbrace{\lim_{h\to 0} \frac{a^h-1}{h}}_{f'(0)}$$

$$\boxed{\frac{d}{dx}\left(a^x\right) = (\ln a)\, a^x}$$

exple $\frac{d}{dx}\left(2^x\right) = (\ln 2)\, 2^x$

by chain Rule: $\boxed{\frac{d}{dx}\, a^{u(x)} = (\ln a)\, u'(x)\, a^{u(x)}}$

exple $\frac{d}{dx}\left(5^{x^2-2x+\sqrt{x}}\right) = (\ln 5) \frac{d}{dx}(x^2-2x+\sqrt{x})\, 5^{x^2-2x+\sqrt{x}}$

$\frac{d}{dx}\, 5^{x^2-2x+\sqrt{x}}$ $= (\ln 5)\left(2x-2+\frac{1}{2\sqrt{x}}\right) 5^{x^2-2x+\sqrt{x}}$

⚠ $\frac{d}{dx}\left(e^x\right) = \underbrace{(\ln e)}_{=1}\, e^x = e^x$

$$\frac{d}{dx}\left(e^{u(x)}\right) = \frac{d}{dx}(u(x)) \cdot e^{u(x)}$$

exple

① $\frac{d}{dx}\left(e^{-3x^4+x^2-1}\right) = \frac{d}{dx}(-3x^4+x^2-1)\, e^{-3x^4+x^2-1}$

$= (-12x^3+2x)\, e^{-3x^4+x^2-1}$

② $\frac{d}{dx}\left(e^x - x\right) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x)$

$= e^x - 1$

③ $\frac{d}{dx}\left(x\, e^x\right) = \frac{d}{dx}(x) \cdot e^x + x \frac{d}{dx}(e^x)$

$= e^x + x e^x$

④ $\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right) = \frac{e^x(1+x^2) - e^x(2x)}{(1+x^2)^2}$

$= \frac{e^x\left[x^2-2x+1\right]}{(1+x^2)^2}$

Derivatives of logarithmic functions

$y = \log_a(x) \iff \underbrace{a^y = x}$

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$$

$$(\ln a)\, y'\, a^y = 1$$

$$y' = \frac{1}{\ln a}\, \frac{1}{a^y}$$

$$\boxed{\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{\ln a}\, \frac{1}{x}} \quad x > 0$$

exple ① $\frac{d}{dx} \log_2(x) = \frac{1}{\ln 2} \cdot \frac{1}{x}$

② $\frac{d}{dx} \log(x) = \frac{1}{\ln 10}\, \frac{1}{x}$

③ $\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{\ln e}\, \frac{1}{x}$

$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$ for $x > 0$.

By chain Rule

$$\boxed{\frac{d}{dx} \log_a |f(x)| = \frac{1}{\ln a}\, \frac{f'(x)}{f(x)}}$$

$$\boxed{\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}}$$

exple Find $f'(x)$ if

① $f(x) = \log_5(x^2-2x+3)$

② $f(x) = \ln\left(x^4 - \sqrt{x} + \frac{1}{x} - 2\right)$

Ans: Using $\frac{d}{dx}\left(\log_a |f(x)|\right) = \frac{1}{\ln a}\, \frac{f'(x)}{f(x)}$

① $f'(x) = \frac{d}{dx}\left(\log_5(x^2-2x+3)\right) = \frac{1}{\ln 5}\, \frac{2x-2}{x^2-2x+3}$

② $f'(x) = \frac{d}{dx}\left(\ln\left(x^4 - \sqrt{x} + \frac{1}{x} - 2\right)\right) = \frac{4x^3 - \frac{1}{2\sqrt{x}} - \frac{1}{x^2}}{x^4 - \sqrt{x} + \frac{1}{x} - 2}$

• Logarithmic differentiation

We use it for a complex function (as a product or division of several functions) to find the derivative

exple $f(x) = (x+1)(x+2)^2(x+3)^3(x+4)^4$

$f'(x) = ?$

Ans We use the logarithmic differentiation

1st step: $\ln|f(x)| = \ln\left|(x+1)(x+2)^2(x+3)^3(x+4)^4\right|$

$= \ln|x+1| + \ln|(x+2)^2| + \ln|(x+3)^3| + \ln|(x+4)^4|$

$\ln|f(x)| = \ln|x+1| + 2\ln|x+2| + 3\ln|x+3| + 4\ln|x+4|$

2nd step we differentiate:

$$\frac{d}{dx} \ln|f(x)| = \frac{d}{dx}\left[\ln|x+1| + 2\ln|x+2| + 3\ln|x+3| + 4\ln|x+4|\right]$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} + \frac{4}{x+4}$$

Finally $f'(x) = \left[\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} + \frac{4}{x+4}\right](x+1)(x+2)^2(x+3)^3(x+4)^4$

exple: Find $f'(x)$ if

① $f(x) = \frac{(x^2+1)^{1/3}(2x+5)^{1/2}}{(3x-1)^2(x^3+1)^4}$

② $f(x) = \left(\sqrt{x+1}\right)^{x^2+1}$

Ans We use logarithm differentiation

① $\ln|f(x)| = \ln\left|\frac{(x^2+1)^{1/3}(2x+5)^{1/2}}{(3x-1)^2(x^3+1)^4}\right|$

$= \ln\left|(x^2+1)^{1/3}(2x+5)^{1/2}\right| - \ln\left|(3x-1)^2(x^3+1)^4\right|$

$= \frac{1}{3}\ln(x^2+1) + \frac{1}{2}\ln|2x+5| - 2\ln|3x-1| - 4\ln|x^3+1|$

Now we differentiate:

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)} = \frac{1}{3}\, \frac{2x}{x^2+1} + \frac{1}{2}\, \frac{2}{2x+5} - 2\, \frac{3}{3x-1} - 4\, \frac{3x^2}{x^3+1}$$

We deduce $f'(x) = \left[\frac{2x}{3(x^2+1)} + \frac{1}{2x+5} - \frac{6}{3x-1} - \frac{12x^2}{x^3+1}\right] f(x)$

② $f(x) = \left(\sqrt{x+1}\right)^{x^2+1} = (x+1)^{\left(\frac{x^2+1}{2}\right)}$

$\ln|f(x)| = \ln\left|(x+1)^{\frac{x^2+1}{2}}\right|$

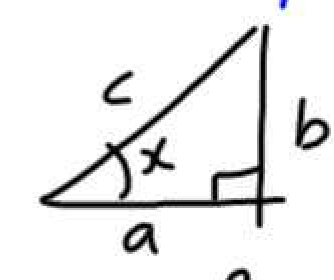
$= \left(\frac{x^2+1}{2}\right)\ln|x+1|$

Now we differentiate,

$$\frac{f'(x)}{f(x)} = x\ln|x+1| + \left(\frac{x^2+1}{2}\right)\frac{1}{x+1}$$

We get $f'(x) = \left[x\ln|x+1| + \frac{x^2+1}{2(x+1)}\right]\left(\sqrt{x+1}\right)^{x^2+1}$

§ Trigonometric functions



pythagor. thm
$c^2 = a^2 + b^2$

We define $\cos x := \frac{a}{c}$; $\sin x = \frac{b}{c}$

⚠ $\cos^2 x + \sin^2 x = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$

$\tan x := \frac{\sin x}{\cos x}$; $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sec x := \frac{1}{\cos x}$; $\csc x := \frac{1}{\sin x}$

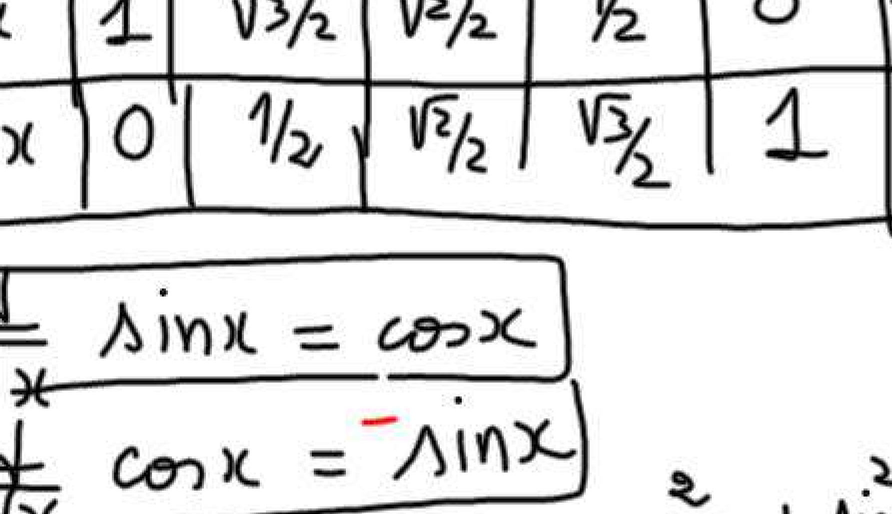
$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$\boxed{1 + \tan^2 x = \sec^2 x}$$

$$1 + \cot^2 x = 1 + \frac{\cos^2 x}{\sin^2 x} = \left(\frac{1}{\sin x}\right)^2$$

$$\boxed{1 + \cot^2 x = \csc^2 x}$$

⚠ $\cos(-x) = \cos x$ (even function)
$\sin(-x) = -\sin x$ (odd function)
$\cos(x + 2\pi) = \cos x$
$\sin(x + 2\pi) = \sin x$ (2π-periodic)



$-1 \le \sin x \le 1$
$-1 \le \cos x \le 1$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	-1
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	0

- $\boxed{\frac{d}{dx} \sin x = \cos x}$
- $\boxed{\frac{d}{dx} \cos x = -\sin x}$
- $\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

- $\frac{d}{dx} \cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$

$$\boxed{\frac{d}{dx} \cot x = -\csc^2 x}$$

- $\frac{d}{dx} \sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = + \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \frac{\sin x}{\cos x}$
$= \sec x \tan x$

$$\boxed{\frac{d}{dx} \sec x = \sec x \tan x}$$

- $\frac{d}{dx} \csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \frac{\cos x}{\sin x}$

$$\boxed{\frac{d}{dx} \csc x = -\csc x \cot x}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad ; -1 < x < 1$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$\cos^2 + \sin^2 = 1$
$\cos^2 = 1 - \sin^2$

قانون الاشتقاق للدوال العكسية $\| \ (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$$(\sin^{-1})'(x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}}$$
$$= \frac{1}{\sqrt{1-x^2}}$$

<u>exple</u> Find $f'(x)$ if:
① $f(x) = \cos(x^2+1)$
② $f(x) = x \sin(\sqrt{x})$
③ $f(x) = 2^{\tan x}$
④ $f(x) = \sec(x^3+1)$
⑤ $f(x) = (\sin(2x))^{\tan^{-1} x}$
⑥ $f(x) = \log_2(\sec(2x))$

<u>Ans</u>:
① $f(x) = \cos(x^2+1)$ then
$f'(x) = -\sin(x^2+1) \frac{d}{dx}(x^2+1)$
$= -2x \sin(x^2+1)$

② $f(x) = x \sin(\sqrt{x})$
$f'(x) = 1 \cdot \sin(\sqrt{x}) + x \cos(\sqrt{x}) \frac{d}{dx}(\sqrt{x})$
$= \sin(\sqrt{x}) + x \cos(\sqrt{x}) \frac{1}{2\sqrt{x}}$

③ $f(x) = 2^{\tan x}$
$f'(x) = (\ln 2)(\sec^2 x) \cdot 2^{\tan x}$

$\| \ \frac{d}{dx} a^{u(x)} = (\ln a)\, u'(x)\, a^{u(x)}$

④ $f(x) = \sec(x^3+1)$
$f'(x) = \sec(x^3+1) \tan(x^3+1)(3x^2)$

⑤ Using logarithm differentiation method
$\ln|f(x)| = \ln\left|(\sin(2x))^{\tan^{-1} x}\right|$
$= (\tan^{-1} x) \cdot \ln|\sin(2x)|$

by differentiation,
$\frac{f'(x)}{f(x)} = \left(\frac{1}{1+x^2}\right) \ln|\sin(2x)| + (\tan^{-1} x) \frac{2\cos(2x)}{\sin(2x)}$

So $f'(x) = \left[\frac{\ln|\sin(2x)|}{1+x^2} + 2\tan^{-1} x \cot(2x)\right] (\sin(2x))^{\tan^{-1} x}$

⑥ $f(x) = \log_2(\sec(2x))$
$f'(x) = \frac{1}{\ln 2} \frac{\sec(2x)\tan(2x) \cdot 2}{\sec(2x)}$
$= \frac{2}{\ln 2} \tan(2x)$

<u>expl</u> find y' if
(75) $y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^2-7)}}$
(76) $y = \frac{e^{x-1} \sin^2 x}{(x^2+5)^{2x}}$

<u>Ans</u>: Using logarithm differentiation
$\ln|y| = \ln\left|\frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}\right|$
$= \ln|e^{2x}(9x-2)^3| - \ln\left((x^2+1)^{1/4}(3x^3-7)^{1/4}\right)$
$= 2x + 3\ln|9x-2| - \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln|3x^3-7|$

Now we differentiate
$\frac{y'}{y} = 2 + 3\frac{9}{9x-2} - \frac{1}{4}\frac{2x}{x^2+1} - \frac{1}{4}\frac{9x^2}{3x^3-7}$

so $y' = \left[2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9}{4}\frac{x^2}{3x^3-7}\right] y$

(76) $y = \frac{e^{x-1} \sin^2 x}{(x^2+5)^{2x}}$
$\ln|y| = \ln\left[e^{x-1}\sin^2 x\right] - \ln\left[(x^2+5)^{2x}\right]$
$= (x-1) + 2\ln|\sin x| - (2x)\ln|x^2+5|$
$\frac{y'}{y} = 1 + 2\frac{\cos x}{\sin x} - 2\ln(x^2+5) - (2x)\left(\frac{2x}{x^2+5}\right)$

Antiderivative functions
(الدوال الأصلية)

Def: We say that F is antiderivative for f on the interval I if
$F'(x) = f(x)$ for every $x \in I$.

exple let $f(x) = 2x$ then
$F(x) = x^2$ is antriderivative on $\mathbb{R}$
Note that $F(x) = x^2+1$ is also an antiderivative of f.

Prop: If F & G be an antiderivative for f on I then there exists $c \in \mathbb{R}$ such that $F(x) - G(x) = c$ for every $x \in I$.

Proof Put $H(x) = F(x) - G(x)$
$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$
So $H(x) = cst$ (constant)

$f(x) =$	antiderivative $F(x) =$
x^r; $r \in \mathbb{R}$	$\frac{x^{r+1}}{r+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{e^{ax}}{a} + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b) + c$
$\sec(ax+b)\tan(ax+b)$	$\frac{1}{a}\sec(ax+b) + c$
$\csc(ax+b)\cot(ax+b)$	$-\frac{1}{a}\csc(ax+b) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$
$\tan x$	$\ln\lvert\sec x\rvert + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$
$-\csc^2 x$	$\cot x + c$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert + c$

⚠ We denote the antiderivative for f by $\boxed{\int f(x)\,dx := F(x) + c}$

---

exple : Compute
1. $\int [2x^3 - 3x^2 + \sqrt{x} + 2]\,dx$
2. $\int e^{-3x}\,dx$
3. $\int \cos(4x-1)\,dx$
4. $\int \left[\frac{1}{x} + \frac{3}{x^2}\right]dx$
5. $\int \sec^2(2x+1)\,dx$

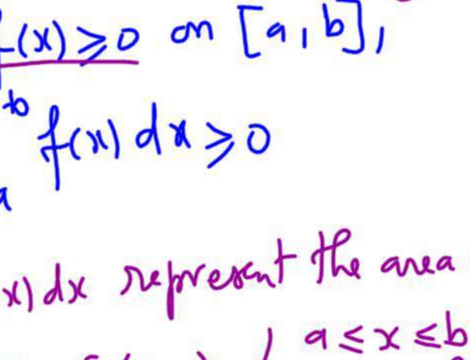
Ans
1. $\int [2x^3 - 2x^2 + \sqrt{x} + 2]\,dx =$
$2\int x^3 dx - 2\int x^2 dx + \int x^{1/2} dx + 2\int 1\,dx =$
$2\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^{3/2}}{3/2} + 2x + c =$
$\frac{1}{2}x^4 - \frac{2}{3}x^3 + \frac{2}{3}x^{3/2} + 2x + c$
2. $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + c$
3. $\int \cos(4x-1)\,dx = \frac{1}{4}\sin(4x-1) + c$
4. $\int \left[\frac{1}{x} + \frac{3}{x^2}\right]dx = \int \frac{1}{x}dx + 3\int \frac{1}{x^2}dx$
$= \ln|x| + 3\int x^{-2}dx$
$= \ln|x| - \frac{3}{x} + c$
5. $\int \sec^2(2x+1)\,dx = \frac{1}{2}\tan(2x+1) + c$

Thm : Let f be a function defined and continuous on $[a,b]$, then f is integrable and
$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$
with $F(x) = \int f(x)\,dx$ is antiderivative for f on $[a,b]$.

exple :
- $\int_0^2 (x^2 - 2x + 1)\,dx = \left[\frac{x^3}{3} - x^2 + x\right]_0^2$
$= \left(\frac{2^3}{3} - 2^2 + 2\right) - (0 - 0 + 0)$
$= \frac{8}{3} - 4 + 2 = \frac{8}{3} - 2 = \frac{2}{3}$
- $\int_0^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_0^{\pi/2}$
$= \sin\frac{\pi}{2} - \sin 0$
$= 1 - 0 = 1$

Prop Let f & g be integrable on $[a,b]$, we have,
1. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
2. $\int_a^b c\,f(x)\,dx = c\left(\int_a^b f(x)\,dx\right)$ for every $c \in \mathbb{R}$.
3. Let $c \in [a,b]$;
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. if $f(x) \geq 0$ on $[a,b]$,
$\int_a^b f(x)\,dx \geq 0$

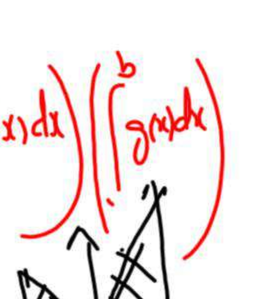
⚠ $\int_a^b f(x)\,dx$ represent the area of the region $R = \left\{(x,y) \,/\, \begin{matrix} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{matrix}\right\}$



$$A(R) = \int_a^b f(x)\,dx.$$

⚠ $\int_a^b f(x)\,g(x)\,dx \neq \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$

example : $\int_{-1}^{2} |x|\,dx = ?$



Ans: $|x| = \begin{cases} x; & \text{if } x \geq 0 \\ (-x); & \text{if } x < 0 \end{cases}$

x	-1	0	2
$\lvert x\rvert =$	$(-x)$	0	x

so $\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_0^2 x\,dx$
$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^2 = \frac{1}{2} + 2 = \frac{5}{2}$

exple Compute

$I = \int_{-1}^{2} (x^2 - 3x)\,dx$; $J = \int_0^{\pi} \sin x\,dx$

$K = \int_{-5}^{-1} \frac{1}{x}\,dx$; $L = \int_0^1 2x e^{x^2}\,dx$

$M = \int_0^4 \frac{2x^2 - 3x + \sqrt{x}}{\sqrt{x}}\,dx$; $N = \int_2^1 \frac{1}{x^2}\,dx$

Ans $I = \int_{-1}^{2} (x^2 - 3x)\,dx = \left[\frac{x^3}{3} - 3\frac{x^2}{2}\right]_{-1}^{2}$

$= \left(\frac{2^3}{3} - \frac{3}{2}2^2\right) - \left(\frac{(-1)^3}{3} - \frac{3}{2}(-1)^2\right)$

$= \left(\frac{8}{3} - 6\right) - \left(-\frac{1}{3} - \frac{3}{2}\right)$

$= -\frac{10}{3} + \frac{1}{3} + \frac{3}{2} = -\frac{3}{2}$

• $J = \int_0^{\pi} \sin x\,dx$

$= -\left[\cos x\right]_0^{\pi} = -\left[\underset{\substack{\parallel \\ 0}}{\cos \pi} - \underset{\substack{\parallel \\ 1}}{\cos 0}\right] = 1$

• $K = \int_{-5}^{-1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-5}^{-1} = \ln|-1| - \ln|-5|$

$= \ln 1 - \ln 5$

$= 0 - \ln 5$

$= -\ln 5$

⚠ $\int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + c$

$L = \int_0^1 2x\, e^{x^2}\,dx = \left[e^{x^2}\right]_0^1 = e^1 - e^0 = e - 1$

$\int u'(x)\, e^{u(x)}\,dx = e^{u(x)} + c$

$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + c$

$M = \int_0^4 \frac{2x^2 - 3x + \sqrt{x}}{\sqrt{x}}\,dx$

$= \int_0^4 (2x^2 - 3x + x^{1/2})\, x^{-1/2}\,dx$

$= \int_0^4 \left(2x^{3/2} - 3x^{1/2} + 1\right) dx$

$= \left[2\frac{x^{5/2}}{5/2} - 3\frac{x^{3/2}}{3/2} + x\right]_0^4$

$= \left[\frac{4}{5}x^{5/2} - 2x^{3/2} + x\right]_0^4$

$= \left(\frac{128}{5} - 2 \cdot 8 + 4\right) - (0 + 0 + 0)$

$= \frac{128}{5} - 12 = \frac{68}{5}$.

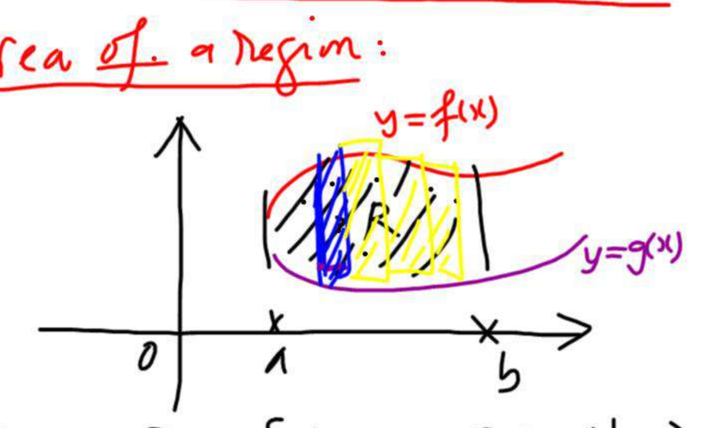
$N = \int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{1} x^{-2}\,dx = \left[-\frac{1}{x}\right]_{-2}^{1}$

$= -\left(1 - \frac{1}{-2}\right) = -1 - \frac{1}{2} = -3/2$.

⚠ if $r \neq -1$; $\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$

---

* Area of a region:



The region $R = \left\{(x,y) \,/\, \substack{a \le x \le b \\ g(x) \le y \le f(x)}\right\}$

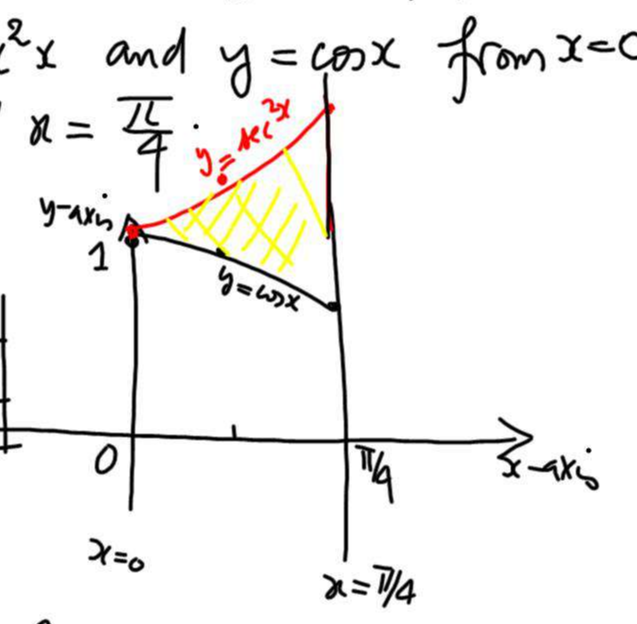
Then the area A is given by

$$A(R) = \int_a^b \left[\underbrace{f(x)}_{\text{upper curve}} - \underbrace{g(x)}_{\text{lower curve.}}\right] dx \geq 0$$

exple: Find the area between the curves $y = \sec^2 x$ and $y = \cos x$ from $x = 0$ and $x = \frac{\pi}{4}$.

Ans

x	0	$\pi/6$	$\pi/4$
$\sec^2 x$	1	$4/3$	2
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\sqrt{2}/2$



$R = \left\{(x,y) \,/\, \substack{0 \le x \le \pi/4 \\ \cos x \le y \le \sec^2 x}\right\}$

So the area of R is

$A(R) = \int_0^{\pi/4} \left[\sec^2 x - \cos x\right] dx$

$= \left[\tan x - \sin x\right]_0^{\pi/4}$

$= \left(1 - \frac{\sqrt{2}}{2}\right) - (0 - 0)$

$= \frac{2 - \sqrt{2}}{2} \geq 0$.

exple 2: Find the area of the region enclosed by $y = (x-1)^2 - 1$ and $y = -x + 2$.

Ans Intersection points

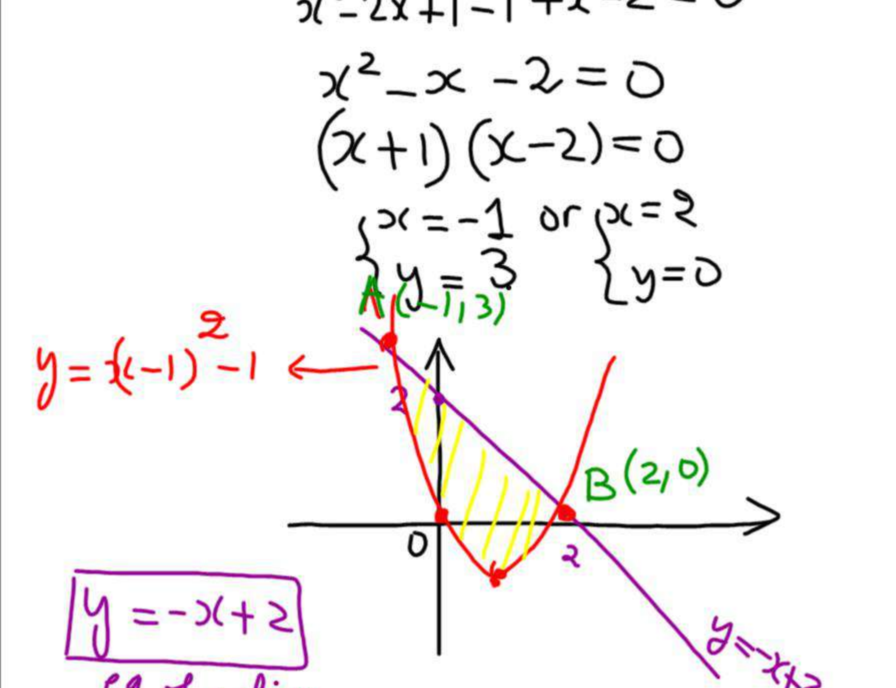
$y = (x-1)^2 - 1 = -x + 2$

$x^2 - 2x + 1 - 1 + x - 2 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\begin{cases} x = -1 \\ y = 3 \end{cases}$ or $\begin{cases} x = 2 \\ y = 0 \end{cases}$



$R = \left\{(x,y) \,/\, \substack{-1 \le x \le 2 \\ (x-1)^2 - 1 \le y \le -x + 2}\right\}$

The area of R is given by:

$A(R) = \int_{-1}^{2} \left[(-x+2) - \left((x-1)^2 - 1\right)\right] dx$

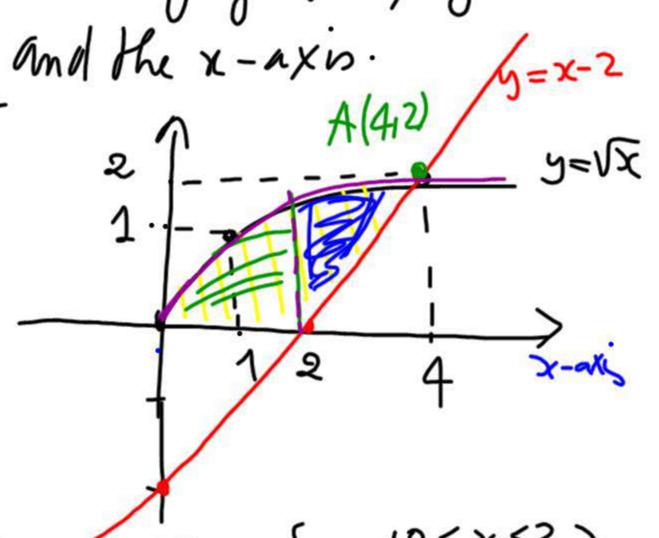
$= \int_{-1}^{2} \left[-x + 2 - \left(x^2 - 2x + \cancel{1} - \cancel{1}\right)\right] dx$

$= \int_{-1}^{2} \left(-x^2 + x + 2\right) dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$

$= \left(-8/3 + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = -3 + \frac{15}{2} = \frac{9}{2}$.

exple Find the area of the region bounded by $y=\sqrt{x}$; $y=x-2$ and the x-axis.

Ans



$R = R_1 \cup R_2 = \left\{(x,y) / \begin{matrix} 0 \le x \le 2 \\ 0 \le y \le \sqrt{x} \end{matrix}\right\} \cup \left\{(x,y) / \begin{matrix} 2 \le x \le 4 \\ x-2 \le y \le \sqrt{x} \end{matrix}\right\}$

Then the area of R is given by:

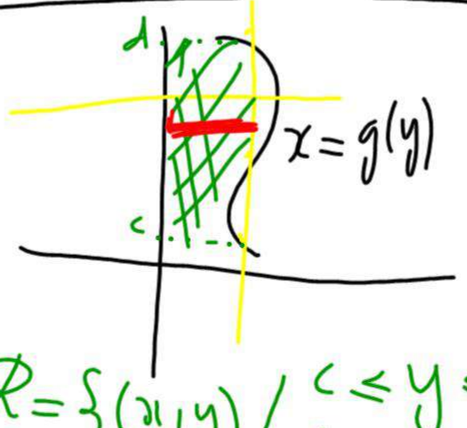
$$A(R) = \int_0^2 \sqrt{x}\, dx + \int_2^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}2^{3/2} + \left(\frac{2}{3}4^{3/2} - 8 + 8\right) - \left(\frac{2}{3}\cdot 2^{3/2} - \frac{2^2}{2} + 4\right)$$

$$= \frac{2}{3}4^{3/2} - 2 = \frac{16}{3} - 2 = \frac{10}{3}.$$

⚠

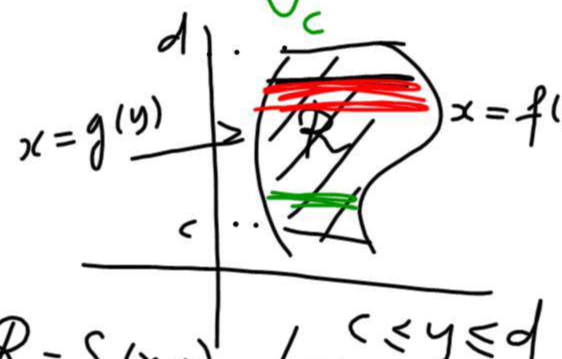


$R = \left\{(x,y) / \begin{matrix} c \le y \le d \\ 0 \le x \le g(y) \end{matrix}\right\}$

The area of R is given by:

$$A(R) = \int_c^d (g(y) - 0)\, dy$$

⚠



$R = \left\{(x,y) / \begin{matrix} c \le y \le d \\ g(y) \le x \le f(y) \end{matrix}\right\}$

Then the area of R is

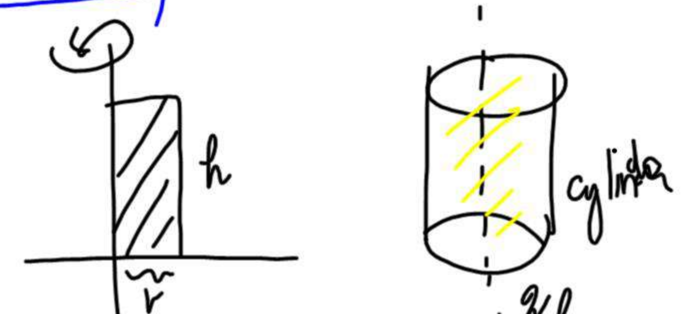
$$A(R) = \int_c^d (f(y) - g(y))\, dy$$

Volume of a Solid

If we revolve a region R about an axis (called axis of revolution) we get a Solid of revolution

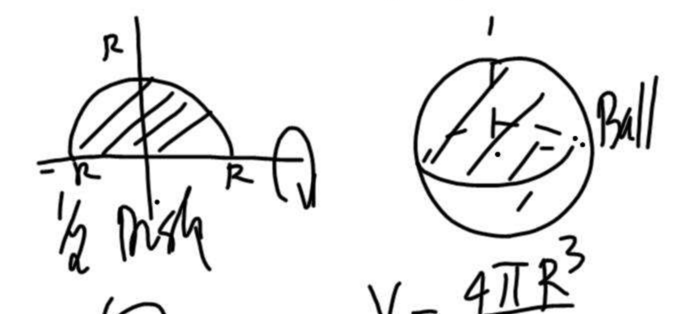
exple

(1)



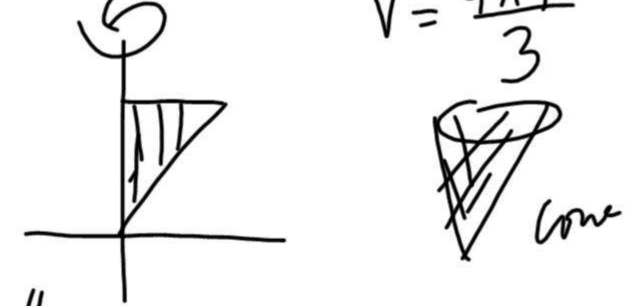
$V = \pi r^2 h$

(2)

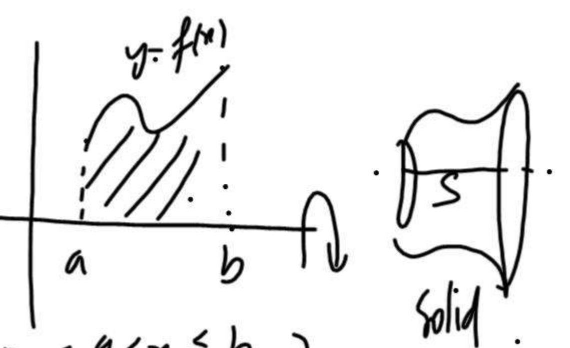


$V = \frac{4\pi R^3}{3}$

(3)



More generally



If we revolve $R = \left\{ \begin{matrix} a \le x \le b \\ 0 \le y \le f(x) \end{matrix} \right\}$ about x-axis we get a solid of revolution S. Its volume is given by (disk method)

$$\boxed{V = \pi \int_a^b (f(x))^2\, dx}$$

example Show that the volume of a ball of radius R is $\frac{4\pi R^3}{3}$

Ans:

If we revolve $\frac{1}{2}$ Disk $= \left\{(x,y) / \begin{matrix} -R \le x \le R \\ 0 \le y \le \sqrt{R^2-x^2} \end{matrix}\right\}$ about x-axis we get a ball of radius R.

Its volume is given by

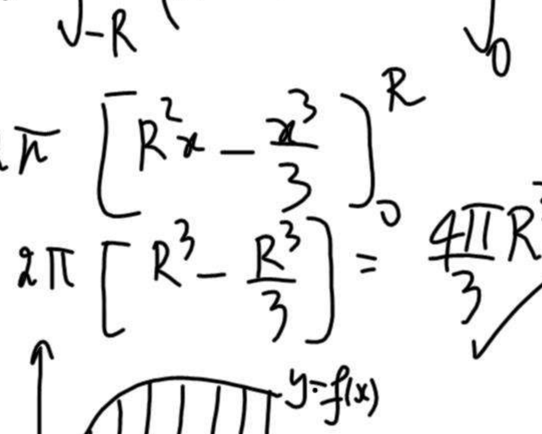
$$V = \pi \int_{-R}^{R} \left(\sqrt{R^2-x^2}\right)^2 dx$$

$$= \pi \int_{-R}^{R} (R^2 - x^2)\, dx = 2\pi \int_0^R (R^2 - x^2)\, dx$$

$$= 2\pi \left[R^2 x - \frac{x^3}{3}\right]_0^R$$

$$= 2\pi \left[R^3 - \frac{R^3}{3}\right] = \frac{4\pi R^3}{3}$$

⚠

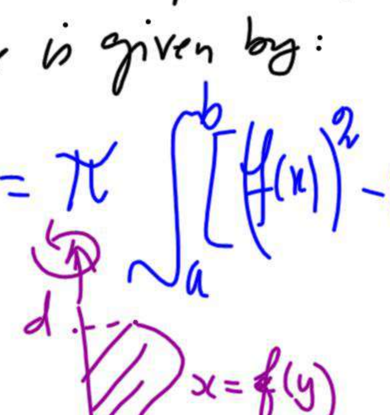


If we revolve $R = \left\{ \begin{matrix} a \le x \le b \\ g(x) \le y \le f(x) \end{matrix} \right\}$ about x-axis, we get a solid S

Its volume is given by:

$$V(S) = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right] dx$$

⚠



If we revolve $R = \left\{(x,y) / \begin{matrix} c \le y \le d \\ 0 \le x \le f(y) \end{matrix}\right\}$ about y-axis

The volume of the solid is given by

$$\boxed{V(S) = \pi \int_c^d (f(y))^2\, dy}$$

§ Integral by substitution

$$\frac{1}{5}\int 5\cos(\boxed{5x+1})\,dx = \frac{1}{5}\int \cos\square\, d\square = \frac{1}{5}\sin(5x+1)+C$$

$$\int \cos\square\, d\square = \sin\square + C$$

We put $u = 5x+1$

$du = 5\,dx$ so $dx = \frac{du}{5}$

$$\int \cos(5x+1)\,dx = \frac{1}{5}\int \cos u\, du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5x+1)+C$$

- $\int x\sin(3x^2+2)\,dx =$

$u = 3x^2+2$

$du = 6x\,dx$ so $x\,dx = \frac{du}{6}$

$$\int x\sin(3x^2+2)\,dx = \frac{1}{6}\int \sin u\, du = -\frac{1}{6}\cos u + C = -\frac{1}{6}\cos(3x^2+2)+C$$

$$\boxed{\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = F(g(x))+C}$$

F antiderivative of f

ex 4 $\int 4x^3(4+x^4)^{1/3}\,dx$

put $u = 4+x^4$

$du = 4x^3\,dx$

So $\int 4x^3(4+x^4)^{1/3}\,dx = \int u^{1/3}\,du = \frac{u^{1/3+1}}{1/3+1}+C = \frac{3}{4}(4+x^4)^{4/3}+C$

ex 8 $\int x\cos(x^2-1)\,dx = \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2-1)+C$

$u = x^2-1$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

ex 11 $\int x\, e^{-x^2/2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2/2}+C$

$u = -x^2/2$

$du = -x\,dx$

$x\,dx = -du$

ex 15 $\int \frac{3x}{x+4}\,dx = 3\int \frac{u-4}{u}\,du$

$u = x+4 \Leftrightarrow x = u-4$

$du = dx$

- $\int \frac{3x}{x+4}\,dx = 3\int\left[1-\frac{4}{u}\right]du = 3\left[u-4\ln|u|\right]+C = 3(x+4)-12\ln|x+4|+C$

ex 18 $I = \int_0^1 (4-x)^{1/7}\,dx = -\int_4^3 u^{1/7}\,du$

$u = (4-x)$; $x=0 \Rightarrow u=4$, $x=1 \Rightarrow u=3$

$du = -dx$

$I = \int_3^4 u^{1/7}\,du = \left[\frac{u^{1+1/7}}{1+1/7}\right]_3^4 = \frac{7}{8}\left(4^{8/7}-3^{8/7}\right)$

$= \frac{7}{8}\left[(4-x)^{8/7}\right]_0^1$

ex 21 $\int \frac{x-1}{1+4x-2x^2}\,dx = -\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|u|+C = -\frac{1}{4}\ln|1+4x-2x^2|+C$

$u = 1+4x-2x^2$

$du = (4-4x)\,dx = -4(x-1)\,dx$

So $(x-1)\,dx = -\frac{1}{4}du$

ex 26 $\int (\cos x)\, e^{\sin x}\,dx = \int e^u\,du = e^u + C = e^{\sin x}+C$

$u = \sin x$

$du = \cos x\,dx$

ex 27 $\int \left(\frac{1}{x}\right)\csc^2(\ln x)\,dx = \int \csc^2 u\,du = -\cot u + C = -\cot(\ln x)+C$

$u = \ln x$

$du = \left(\frac{1}{x}dx\right)$

ex 31 $\int \tan x\sec^2 x\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\tan^2 x}{2}+C$

$u = \tan x$

$du = \sec^2 x\,dx$

ex 33 $\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3}+C = \frac{(\ln x)^3}{3}+C$

$u = \ln x$

$du = \frac{1}{x}dx$

ex 35 $I = \int x^3\sqrt{5+x^2}\,dx$

$= \int x^3(5+x^2)^{1/2}\,dx = \frac{1}{2}\int (u-5)\,u^{1/2}\,du$

$u = 5+x^2$

$du = 2x\,dx$

$x^3\,dx = x^2\, x\,dx = \frac{1}{2}(u-5)\,du$

$I = \int x^3(5+x^2)^{1/2}\,dx = \frac{1}{2}\int\left[u^{3/2}-5u^{1/2}\right]du$

$= \frac{1}{2}\left[\frac{2}{5}u^{5/2}-5\frac{2}{3}u^{3/2}\right]+C$

$= \frac{1}{5}u^{5/2}-\frac{5}{3}u^{3/2}+C$

$= \frac{1}{5}(5+x^2)^{5/2}-\frac{5}{3}(5+$

§ Integration by Parts

تكامل بالتجزئة

$$(u(x)\cdot v(x))' = u'(x)v(x) + u(x)v'(x)$$

$$u(x)v'(x) = (u(x)v(x))' - v(x)u'(x)$$

قانون التكامل بالتجزئة

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx}$$

- $\int \underbrace{x}\,\underbrace{\cos x}\,dx = x\sin x - \int \sin x\,dx$

$u(x) = x$, $v'(x) = \cos x$ $\Rightarrow$ $u'(x) = 1$, $v(x) = \sin x$

$$\boxed{\int x\cos x\,dx = x\sin x + \cos x + C}$$

Check $\frac{d}{dx}\left[x\sin x + \cos x + c\right] = \cancel{\sin x} + x\cos x - \cancel{\sin x} = x\cos x$

- $\int \underbrace{x}\,\underbrace{e^x}\,dx = x e^x - \int e^x\,dx$

$u(x) = x$, $V'(x) = e^x$ $\Rightarrow$ $u'(x) = 1$, $V(x) = e^x$

$$\int x e^x\,dx = x e^x - e^x + c = (x-1)e^x + c$$

- $\int \underbrace{x^2}\,e^x\,dx = x^2 e^x - 2\int x e^x\,dx$

$u(x) = x^2$, $V'(x) = e^x$ $\Rightarrow$ $u'(x) = 2x$, $V(x) = e^x$

$$\int x^2 e^x\,dx = x^2 e^x - (x-1)e^x + c = (x^2 - x + 1)e^x + c$$

- $\int 1.\ln x\,dx = x\ln x - \int 1\,dx$

$u(x) = \ln x$, $V'(x) = 1$ $\Rightarrow$ $u'(x) = \frac{1}{x}$, $V(x) = x$

$$\int \ln x\,dx = x\ln x - x + c$$

Check $\frac{d}{dx}\left[x\ln x - x + c\right] = \ln x + \cancel{x\cdot\frac{1}{x}} \cancel{-1}$

- $\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx$

$u(x) = \ln x$, $V'(x) = x^3$ $\Rightarrow$ $u'(x) = \frac{1}{x}$, $V(x) = \frac{x^4}{4}$

So $\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + c$

- $I = \int e^x \sin x\,dx = e^x \sin x - \underbrace{\int e^x \cos x\,dx}_{J}$

$u(x) = \sin x$, $V'(x) = e^x$ $\Rightarrow$ $u'(x) = \cos x$, $V(x) = e^x$

$J = \int e^x \cos x\,dx = e^x \cos x + \underbrace{\int e^x \sin x\,dx}_{= I}$

$u_1(x) = \cos x$, $V_1'(x) = e^x$ $\Rightarrow$ $u_1'(x) = -\sin x$, $V_1(x) = e^x$

$$I = e^x \sin x - e^x \cos x - I$$

$$2I = e^x[\sin x - \cos x]$$

$$I = \int e^x \sin x\,dx = \frac{1}{2}e^x[\sin x - \cos x] + C$$

- $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$u = \tan^{-1}x$, $V'(x) = 1$ $\Rightarrow$ $u'(x) = \frac{1}{1+x^2}$, $V(x) = x$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

- $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$u(x) = \sin^{-1}x$, $V'(x) = 1$ $\Rightarrow$ $u'(x) = \frac{1}{\sqrt{1-x^2}}$, $V(x) = x$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

ex 2 $\int 3x\cos x\,dx = 3\int x\cos x\,dx = 3\left[x\sin x - \int \sin x\,dx\right]$

$u(x) = x$, $V'(x) = \cos x$ $\Rightarrow$ $u'(x) = 1$, $V(x) = \sin x$

$$\int 3x\cos x\,dx = 3x\sin x + 3\cos x + C$$

ex 6 $\int x\sin(1-2x)\,dx = \frac{x}{2}\cos(1-2x) - \frac{1}{2}\int \cos(1-2x)\,dx$

$u(x) = x$, $V'(x) = \sin(1-2x)$ $\Rightarrow$ $u'(x) = 1$, $V(x) = -\frac{\cos(1-2x)}{-2}$

$$\int x\sin(1-2x)\,dx = \frac{x}{2}\cos(1-2x) + \frac{1}{4}\sin(1-2x) + C$$

ex 10 $\int 2x^2 e^{-x}\,dx = 2\left(\int x^2 e^{-x}\,dx\right)$

$u = x^2$, $V' = e^{-x}$ $\Rightarrow$ $u' = 2x$, $V = -e^{-x}$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

$u_1 = x$, $V_1' = e^{-x}$ $\Rightarrow$ $u_1' = 1$, $V_1 = -e^{-x}$

$$\int x e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx = -x e^{-x} - e^{-x} + c$$

$$\int 2x^2 e^{-x}\,dx = -2x^2 e^{-x} + 4(-x-1)e^{-x} + c = -2\left[x^2 + 2x + 2\right]e^{-x} + c$$

ex 11 $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

$u = \ln x$, $V' = x$ $\Rightarrow$ $u' = 1/x$, $V = x^2/2$

ex12 $\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$

$u(x) = \ln x \qquad u'(x) = \frac{1}{x}$
$\Rightarrow$
$V'(x) = x^2 \qquad V(x) = \frac{x^3}{3}$

$\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + c$

ex15

$\int x \sec^2 x \, dx = x\tan x - \int \tan x \, dx$

$u(x) = x \qquad u'(x) = 1$
$\Rightarrow$
$V'(x) = \sec^2 x \qquad V(x) = \tan x$

$\int x \sec^2 x \, dx = x\tan x - \ln|\sec x| + C$

ex19 $\int_1^2 \ln x \, dx = [x \ln x]_1^2 - \underbrace{\int_1^2 1 \, dx}_{[x]_1^2}$

$U(x) = \ln x \qquad u'(x) = 1/x$
$\Rightarrow$
$V'(x) = 1 \qquad V(x) = x$

$\int_1^2 \ln x \, dx = 2\ln 2 - 1$

ex20 $\int_1^e \ln(x^2) dx = 2\int_1^e (\ln x) dx$

$= 2[x \ln x]_1^e - [x]_1^e$

$= 2(e - 0) - (e-1) = e+1$

ex31

$I = \int \cos^2 x \, dx = \int \cos x \cos x \, dx$

$u(x) = \cos x \qquad u'(x) = -\sin x$
$\Rightarrow$
$V'(x) = \cos x \qquad V(x) = \sin x$

$\int \cos^2 x \, dx = \cos x \sin x + \int \underbrace{\sin^2 x}_{\sin^2 x = 1 - \cos^2 x} dx$

$= \cos x \sin x + \int [1 - \cos^2 x] dx$

$\int \cos^2 x \, dx = \cos x \sin x + x - \int \cos^2 x \, dx$

$2\int \cos^2 x \, dx = \cos x \sin x + x + c$

$$\boxed{\int \cos^2 x \, dx = \frac{1}{2}\cos x \sin x + \frac{x}{2} + cst}$$

⚠ second method

$\cos^2 x = \frac{1+\cos 2x}{2}$

So $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

$= \frac{x}{2} + \frac{\cancel{2}\sin x \cos x}{\cancel{4}_2} + c$

§ Integrals of Rational Functions (Partial Fractions) (7.3.1, 7.3.2)

$$\int \frac{P(x)}{Q(x)}\,dx$$

- if $\deg P \geq \deg Q$, we do long division

$$\frac{P(x)}{Q(x)} = E(x) + \frac{P_1(x)}{Q(x)} \qquad \deg P_1 < \deg Q$$

$E(x)$: polynomial of degree $= \deg P - \deg Q$

- Using Partial fractions decomposition to evaluate $\int \frac{P_1(x)}{Q(x)}\,dx$

exple 1 $\displaystyle\int \frac{x}{x+2}\,dx$; $\frac{x}{x+2} = \frac{P(x)}{Q(x)}$, $\deg P = 1 = \deg Q$

long division

For $x \neq -2$

$$\begin{array}{r|l} x & x+2 \\ \hline -x-2 & 1 \\ \hline -2 & \end{array}$$

$$\frac{x}{x+2} = 1 - \frac{2}{x+2}$$

So
$$\int \frac{x}{x+2}\,dx = \int \left[1 - \frac{2}{x+2}\right]dx = x - 2\int \frac{dx}{x+2} = x - 2\ln|x+2| + c$$

exple 2 Evaluate $\displaystyle\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5}\,dx$

Ans: • Long division

$$\begin{array}{r|l} 3x^3-7x^2+17x-3 & x^2-2x+5 \\ \hline -3x^3+6x^2-15x & 3x-1 \\ \hline -x^2+2x-3 & \\ x^2-2x+5 & \\ \hline 2 & \end{array}$$

$$\frac{3x^3-7x^2+17x-3}{x^2-2x+5} = (3x-1) + \frac{2}{x^2-2x+5}$$

So
$$\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5}\,dx = \int\left[(3x-1) + \frac{2}{x^2-2x+5}\right]dx$$
$$= 3\frac{x^2}{2} - x + 2\int \frac{dx}{x^2-2x+5}$$

$$\square^2 + 2\square\Delta = (\square+\Delta)^2 - \Delta^2$$
$$\square^2 - 2\square\Delta = (\square-\Delta)^2 - \Delta^2$$
$$x^2 - 2x = (x-1)^2 - 1^2$$
$$x^2-2x+5 = (x-1)^2 - 1 + 5 = (x-1)^2 + 4$$

$$\int \frac{dx}{x^2-2x+5} = \int \frac{dx}{\underbrace{2}_{a}^2 + \underbrace{(x-1)}_{u}^2} = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + c$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5}\,dx = \frac{3x^2}{2} - x + \tan^{-1}\left(\frac{x-1}{2}\right) + c$$

- Partial fraction decomposition

exp 3
$$\int \frac{dx}{x(x-1)} = \int\left[\frac{A}{x} + \frac{B}{x-1}\right]dx = A\int\frac{dx}{x} + B\int\frac{dx}{x-1} = A\ln|x| + B\ln|x-1| + c$$

$A = -1$, $B = 1$

$$f(x) = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$A = \lim_{x\to 0} x f(x) = \lim_{x\to 0}\frac{1}{x-1} = -1$$
$$B = \lim_{x\to 1}(x-1)f(x) = \lim_{x\to 1}\frac{1}{x} = 1$$

$$\int \frac{dx}{x(x-1)} = -\ln|x| + \ln|x-1| + c$$

exple 4
$$\int \frac{x}{(x+1)^2}\,dx = \int\left[\frac{A}{x+1} + \frac{B}{(x+1)^2}\right]dx = A\left(\int\frac{dx}{x+1}\right) + B\left(\int\frac{dx}{(x+1)^2}\right) = A\ln|x+1| - \frac{B}{x+1} + c$$

$A = 1$, $B = -1$

$$f(x) = \frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

So $x = A(x+1) + B = Ax + (A+B)$

$$\begin{cases} 1 = A \\ 0 = A+B \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\int \frac{x}{(x+1)^2}\,dx = \ln|x+1| + \frac{1}{x+1} + c$$

exple 5

•
$$\int \frac{dx}{x^2(x+1)} = \int\left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}\right]dx = A\left(\int\frac{dx}{x}\right) + B\left(\int\frac{dx}{x^2}\right) + C\left(\int\frac{dx}{x+1}\right)$$
$$= A\ln|x| - \frac{B}{x} + C\ln|x+1| + \text{cst}$$

$$f(x) = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}; \quad x \neq 0,\ x \neq -1$$

$$B = \lim_{x\to 0} x^2 f(x) = \lim_{x\to 0}\frac{1}{x+1} = 1$$
$$C = \lim_{x\to -1}(x+1)f(x) = \lim_{x\to -1}\frac{1}{x^2} = 1$$

$$f(x) = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$f(1) = \frac{1}{2} = A + 1 + \frac{1}{2} \quad \text{so } \boxed{A = -1}$$

$$\int \frac{dx}{x^2(x+1)} = -\ln|x| - \frac{1}{x} + \ln|1+x| + \text{cst}$$

exple 6:
$$I = \int \frac{2x^3-x^2+2x-2}{(x^2+2)(x^2+1)}\,dx = \int\frac{P(x)}{Q(x)}\,dx$$

$\deg P = 3$
$\deg Q = 4$

$$= \int\left[\frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}\right]dx$$
$$= \int\frac{Ax+B}{x^2+2}\,dx + \int\frac{Cx+D}{x^2+1}\,dx$$

$$\boxed{\int\frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c}$$

$$= \frac{A}{2}\int\frac{2x}{x^2+2}\,dx + B\int\frac{dx}{x^2+2} + \frac{C}{2}\int\frac{2x}{x^2+1}\,dx + D\int\frac{dx}{1+x^2}$$

$$= \frac{A}{2}\ln(x^2+2) + \frac{B}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{C}{2}\ln(1+x^2) + D\tan^{-1}(x) + c$$

Finding A, B, C, D

$$f(x) = \frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{(Ax+B)(x^2+1) + (Cx+D)(x^2+2)}{(x^2+2)(x^2+1)}$$

So $2x^3 - x^2 + 2x - 2 = (Ax+B)(x^2+1) + (Cx+D)(x^2+2)$
$$= Ax^3 + Ax + Bx^2 + B + Cx^3 + 2Cx + Dx^2 + 2D$$
$$= (A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D)$$

$$\begin{cases} (1)\ A + C = 2 \\ (2)\ B + D = -1 \\ (3)\ A + 2C = 2 \\ (4)\ B + 2D = -2 \end{cases}$$

$(3) - (1) \Rightarrow C = 0 \Rightarrow A = 2$

$(4) - (2) \Rightarrow D = -1 \Rightarrow B = 0$

$$\int \frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} dx = \ln(x^2+2) - \tan^{-1}x + cst$$

expl 7 $$J = \int \frac{x^2+x+1}{(x^2+1)^2} dx$$
$$= \int \left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \right] dx$$
$$= \frac{A}{2} \int \frac{2x}{x^2+1} dx + B \int \frac{dx}{x^2+1} + \frac{C}{2} \int \frac{2x}{(x^2+1)^2} dx + D \int \frac{dx}{(x^2+1)^2}$$
$$= \frac{A}{2} \ln(1+x^2) + B \tan^{-1}x - \frac{C}{2} \frac{1}{1+x^2} + D \int \frac{dx}{(x^2+1)^2}$$

Finding the constants A, B, C and D

$$f(x) = \frac{x^2+x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$
$$= \frac{(Ax+B)(x^2+1) + Cx + D}{(x^2+1)^2}$$

So $x^2 + x + 1 = (Ax+B)(x^2+1) + Cx + D$
$$= Ax^3 + Ax + Bx^2 + B + Cx + D$$
$$= Ax^3 + Bx^2 + (A+C)x + (B+D)$$

$$\begin{cases} A = 0 \\ B = 1 \\ A + C = 1 \\ B + D = 1 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = 1 \\ C = 1 \\ D = 0 \end{cases}$$

$$J = \int \frac{x^2+x+1}{(x^2+1)^2} dx = \tan^{-1}x - \frac{1}{2} \frac{1}{1+x^2} + cst.$$

ex 1 page 350

$$I = \int \frac{2x^2+5x-1}{x+2} dx$$

long division $\longrightarrow$

$$\begin{array}{r|l} 2x^2+5x-1 & x+2 \\ \underline{-2x^2-4x} & 2x+1 \\ x-1 & \\ \underline{-x-2} & \\ -3 & \end{array}$$

$$I = \int \left[(2x+1) - \frac{3}{x+2} \right] dx$$
$$= x^2 + x - 3 \ln|x+2| + cst$$

Ex 3

$$J = \int \frac{3x^3 + 5x - 2x^2 - 2}{x^2+1} dx$$

Using long division

$$\begin{array}{r|l} 3x^3 + 5x - 2x^2 - 2 & x^2+1 \\ \underline{-3x^3 - 3x} & 3x-2 \\ -2x^2 + 2x - 2 & \\ \underline{2x^2 + 2} & \\ 2x & \end{array}$$

$$J = \int \frac{3x^3 + 5x - 2x^2 - 2}{x^2+1} dx = \int \left[(3x-2) + \frac{2x}{x^2+1} \right] dx$$
$$= \frac{3x^2}{2} - 2x + \ln(1+x^2) + c$$

ex 7

$$K = \int \frac{4x^2 - 14x - 6}{x(x-3)(x+1)} dx$$
$$= \int \left[\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1} \right] dx$$
$$= A \ln|x| + B \ln|x-3| + C \ln|x+1| + cst$$

$2 = A$, $-1 = B$, $C = 3$

$$f(x) = \frac{4x^2 - 14x - 6}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}$$

- $A = \lim_{x \to 0} x f(x) = \lim_{x \to 0} \frac{4x^2 - 14x - 6}{(x-3)(x+1)} = \frac{-6}{-3} = 2$
- $B = \lim_{x \to 3} (x-3) f(x) = \lim_{x \to 3} \frac{4x^2 - 14x - 6}{x(x+1)} = \frac{36-42-6}{12} = -1$
- $C = \lim_{x \to -1} (x+1) f(x) = \lim_{x \to -1} \frac{4x^2 - 14x - 6}{x(x-3)} = \frac{4+14-6}{4} = 3$

ex 8

$$L = \int \frac{16x - 6}{(2x-5)(3x+1)} dx$$
$$= \int \left[\frac{A}{2x-5} + \frac{B}{3x+1} \right] dx$$
$$= \frac{A}{2} \int \frac{2\,dx}{2x-5} + \frac{B}{3} \int \frac{3\,dx}{3x+1}$$
$$= \frac{A}{2} \ln|2x-5| + \frac{B}{3} \ln|3x+1| + cst$$

$A = 4$, $B = 2$

Finding A & B

$$f(x) = \frac{16x-6}{(2x-5)(3x+1)} = \frac{A}{2x-5} + \frac{B}{3x+1}$$

So
- $A = \lim_{x \to 5/2} (2x-5) f(x) = \lim_{x \to 5/2} \frac{16x-6}{3x+1} = \frac{40-6}{\frac{15}{2}+1} = \frac{34 \times 2}{17} = 4$
- $B = \lim_{x \to -1/3} (3x+1) f(x) = \lim_{x \to -1/3} \frac{16x-6}{2x-5} = \frac{-16/3 - 6}{-2/3 - 5} = \frac{-34}{-17} = 2$

$$L = \int \frac{16x-6}{(2x-5)(3x+1)} dx$$
$$= 2 \ln|2x-5| + \frac{2}{3} \ln|3x+1| + cst$$

ex 9

$$\int \frac{5x-1}{x^2-1} dx = \int \frac{5x-1}{(x-1)(x+1)} dx$$

⚠ $x^2 - 1 = (x-1)(x+1)$ $$= \int \left[\frac{A}{x-1} + \frac{B}{x+1} \right] dx$$

$$\int \frac{5x-1}{x^2-1}\,dx = A\int\frac{dx}{x-1} + B\int\frac{dx}{x+1}$$
$$= A\ln|x-1| + B\ln|x+1| + cst$$

2 3

$$f(x) = \frac{5x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1};\ x \neq \pm 1$$

So $A = \lim_{x\to 1}(x-1)f(x) = \lim_{x\to 1}\frac{5x-1}{x+1} = 2$

$B = \lim_{x\to -1}(x+1)f(x) = \lim_{x\to -1}\frac{5x-1}{x-1} = 3$

ex 11

$$M = \int \frac{4x+1}{x^2-3x-10}\,dx$$

$x^2-3x-10 = 0$ $\qquad ax^2+bx+c = a(x-x_1)(x-x_2)$

$a=1;\ b=-3;\ c=-10$

discriminant $D = b^2-4ac = 9+40 = 49 = 7^2$

$x_1 = \frac{3-7}{2} = -2$

$x_2 = \frac{3+7}{2} = 5$

$x^2-3x-10 = (x+2)(x-5)$

$$M = \int\frac{4x+1}{x^2-3x-10}\,dx = \int\frac{(4x+1)}{(x+2)(x-5)}\,dx$$
$$= \int\left[\frac{A}{x+2} + \frac{B}{x-5}\right]dx$$

1 3
$$= A\ln|x+2| + B\ln|x-5| + cst$$

$$f(x) = \frac{4x+1}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}$$

- $A = \lim_{x\to -2}(x+2)f(x) = \lim_{x\to -2}\frac{4x+1}{x-5} = \frac{-7}{-7} = 1$
- $B = \lim_{x\to 5}(x-5)f(x) = \lim_{x\to 5}\frac{4x+1}{x+2} = \frac{21}{7} = 3$

ex 13:
$$\int\frac{dx}{x(x-2)} = \int\left[\frac{A}{x} + \frac{B}{x-2}\right]dx$$
$$= A\ln|x| + B\ln|x-2| + cst$$

$$f(x) = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$A = \lim_{x\to 0} x\,f(x) = \lim_{x\to 0}\frac{1}{x-2} = -\frac{1}{2}$

$B = \lim_{x\to 2}(x-2)f(x) = \lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$

ex 16:
$$\int\frac{dx}{(x-1)(x+2)} = \int\left[\frac{A}{x-1} + \frac{B}{x+2}\right]dx$$
$$= A\ln|x-1| + B\ln|x+2| + cst$$

$A = ?$ $\quad B = ?$

ex 18

$$I = \int\frac{4x^2-x-1}{(x+1)^2(x-3)}\,dx = \int\left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}\right]$$
$$= A\ln|x+1| - \frac{B}{x+1} + C\ln|x-3| + cst$$

Finding A, B & C:

$$f(x) = \frac{4x^2-x-1}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

- $B = \lim_{x\to -1}(x+1)^2 f(x) = \lim_{x\to -1}\frac{4x^2-x-1}{x-3} = \frac{4}{-4} = -1$
- $C = \lim_{x\to 3}(x-3)f(x) = \lim_{x\to 3}\frac{4x^2-x-1}{(x+1)^2} = \frac{32}{16} = 2$
- $f(0) = \frac{-1}{-3} = A - 1 + \frac{2}{-3}$

$A = \frac{1}{3} + 1 + \frac{2}{3} = 2$

ex 20

$$N = \int\frac{x^3-3x^2-x-6}{(x^2+2)(x^2+1)}\,dx$$
$$= \int\left[\frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}\right]dx$$
$$= \frac{A}{2}\int\frac{2x}{x^2+2}\,dx + B\int\frac{dx}{(\sqrt{2})^2+x^2} + \frac{C}{2}\int\frac{2x}{x^2+1}\,dx + D\int\frac{dx}{1+x^2}$$
$$= \frac{A}{2}\ln(x^2+2) + \frac{B}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{C}{2}\ln(1+x^2) + D\tan^{-1}x + cst$$

Finding A, B, C & D:

$$f(x) = \frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{(Ax+B)(x^2+1)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+1)}$$

So $x^3-3x^2+x-6 = (Ax+B)(x^2+1) + (Cx+D)(x^2+2)$
$$= Ax^3+Ax+Bx^2+B+Cx^3+2Cx+Dx^2+2D$$
$$= (A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D)$$

$$\begin{cases} A+C = 1 & (1)\\ B+D = -3 & (2)\\ A+2C = 1 & (3)\\ B+2D = -6 & (4)\end{cases}$$

$(3)-(1) \Rightarrow C = 0 \overset{(1)}{\Rightarrow} A = 1$

$(4)-(2) \Rightarrow D = -3 \overset{(2)}{\Rightarrow} B = 0$

$$\int\frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)}\,dx = \frac{1}{2}\ln(x^2+2) - 3\tan^{-1}x + cst$$

§ Improper Integral (7.4.1, 7.4.2)

§ Improper Integrals

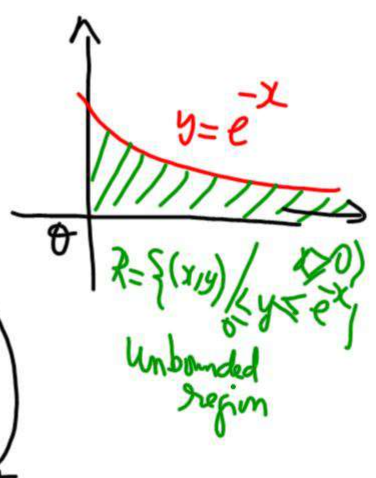
- Unbounded Interval

$$\int_0^{+\infty} e^{-x}\,dx \;;\quad \int_1^{\infty} \frac{1}{x^2}\,dx \;;\quad \int_1^{\infty} \frac{dx}{\sqrt{x}}$$

- Unbounded Integrand

$$\int_0^1 \frac{dx}{\sqrt{x}} \;;\quad \int_0^1 \frac{dx}{(x-1)^{2/3}} \;;\quad \int_0^1 \ln x\,dx$$

§ Unbounded Interval:



expl $I = \int_0^{+\infty} e^{-x}\,dx$

$$= \lim_{t\to+\infty}\left(\int_0^t e^{-x}\,dx\right)$$
$$= \lim_{t\to+\infty}\left[-e^{-x}\right]_0^t$$
$$= \lim_{t\to+\infty}\left[-e^{-t}+e^{0}\right]$$
$$= -e^{-\infty}+1 = 1$$

We say that the improper integral $\int_0^{+\infty} e^{-x}\,dx$ is Convergent to 1.

expl show that:
$\int_1^{\infty} \frac{dx}{x^p}$ is convergent if and only if $p>1$

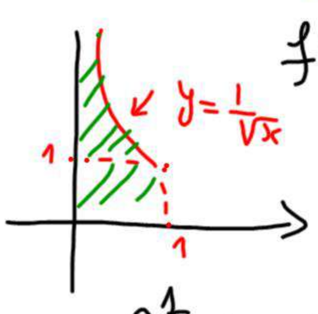
Ans
- $f(x)=\frac{1}{x^p}$ is continuous on $[1,+\infty)$
- If $\boxed{p=1}$; $\int_1^{\infty}\frac{dx}{x} = \lim_{t\to+\infty}\left(\int_1^t \frac{dx}{x}\right)$
$$= \lim_{t\to+\infty}\left[\ln|x|\right]_1^t$$
$$= \lim_{t\to+\infty}\left[\ln|t| - \underbrace{\ln 1}_{0}\right]$$
$$= \ln^{+}\infty = +\infty$$
So $\left(\int_1^{+\infty}\frac{dx}{x}\right)$ is divergent.
- if $p\neq 1$,
$$\int_1^{+\infty}\frac{dx}{x^p} = \lim_{t\to+\infty}\left(\int_1^t x^{-p}\,dx\right)$$
$$= \lim_{t\to+\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$$
$$= \frac{1}{1-p}\lim_{t\to+\infty}\left[t^{1-p}-1\right]$$
$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases}$$

We deduce $\int_1^{+\infty}\frac{dx}{x^p}$ is convergent to $\frac{1}{p-1}$ if and only if $p>1$.

Appl
- $\left(\int_1^{+\infty}\frac{dx}{x^2}\right)$ is convergent because $p=2>1$
- $\left(\int_1^{+\infty}\frac{dx}{\sqrt{x}}\right)$ is divergent because $p=\frac{1}{2}<1$

- Unbounded Integrand

expl $\int_0^1 \frac{dx}{\sqrt{x}}$ is an improper Integral



$f(x)=\frac{1}{\sqrt{x}}$ is not bounded at $x=0$ and not defined
$\lim_{x\to 0^+} f(x)=\infty$

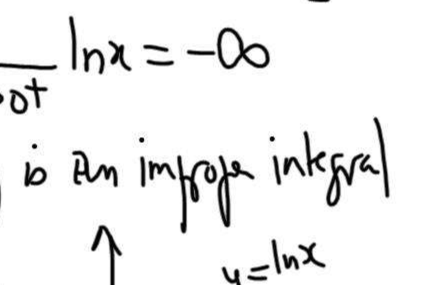
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(2\int_t^1 \frac{dx}{2\sqrt{x}}\right)$$
$$= 2\lim_{t\to 0^+}\left[\sqrt{x}\right]_t^1$$
$$= 2\lim_{t\to 0^+}\left[\sqrt{1}-\underbrace{\sqrt{t}}_{0}\right]$$
$$= 2$$

So $\left(\int_0^1\frac{dx}{\sqrt{x}}\right)$ is convergent to 2

expl Determine if the following integral $\int_0^1 \ln x\,dx$ is convergent or divergent

Ans $f(x)=\ln x$ is continuous on $(0,1]$
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\ln x = -\infty$
So $\left(\int_0^1 \ln x\,dx\right)$ is an improper integral

$$\int_0^1 (\ln x)\,dx = \lim_{t\to 0^+}\left(\int_t^1 \ln x\,dx\right)$$



$$\int_t^1 \ln x\,dx = \left[x\ln x\right]_t^1 - \int_t^1 \frac{1}{x}\cdot x\,dx$$

$u(x)=\ln x$, $V'(x)=1$ $\Rightarrow$ $u'(x)=1/x$, $V(x)=x$

$$\int_t^1 \ln x\,dx = \left[x\ln x - x\right]_t^1 = (-1) - t\ln t + t$$
$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} t-1-t\ln t = -1$$

because $\lim_{t\to 0^+} t\ln t = 0$

So $\left(\int_0^1 \ln x\,dx\right)$ is convergent to (-1).

exercises 7.4.1, 7.4.2) page 362
ex 1, 4, 8, 9, 12, 16, 17, 19, 23, 25

① $\int_0^{+\infty} 3e^{-6x}\,dx = \lim_{t\to+\infty}\left(\int_0^t 3e^{-6x}\,dx\right)$
$$= \lim_{t\to\infty} 3\left[\frac{e^{-6x}}{-6}\right]_0^t$$
$$= -\frac{1}{2}\lim_{t\to+\infty}\left(e^{-6t}-1\right)$$
$$= 1/2 \quad \text{because } e^{-\infty}=0$$

④ $\int_e^{+\infty}\frac{dx}{x(\ln x)^2} = \lim_{t\to+\infty}\left(\int_e^t \frac{dx}{x(\ln x)^2}\right)$
$$= \lim_{t\to+\infty}\left(\int_{\ln e}^{\ln t}\frac{dv}{v^2}\right)$$
$v=\ln x$, $dv=\frac{dx}{x}$
$$= \lim_{t\to+\infty}\left[-\frac{1}{v}\right]_{\ln e}^{\ln t}$$
$$= -\lim_{t\to+\infty}\left[\frac{1}{\ln t}-1\right] = 1$$
because $\ln^{+}\infty = +\infty$, $\ln e = 1$

(8) $\displaystyle\int_{-\infty}^{+\infty} x e^{-x^2/2} dx = \int_{-\infty}^{0} x e^{-x^2/2} dx + \int_{0}^{+\infty} x e^{-x^2/2} dx$

$\displaystyle = \lim_{u \to -\infty} \left(\int_u^0 x e^{-x^2/2} dx \right) + \lim_{t \to +\infty} \left(\int_0^t x e^{-x^2/2} dx \right)$

$\displaystyle = \lim_{u \to -\infty} -\left[e^{-x^2/2} \right]_u^0 + \lim_{t \to +\infty} -\left[e^{-x^2/2} \right]_0^t$

$= -[1-0] + -[0-1]$

$= -1 + 1 = 0$

(9) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2} dx = \int_{-\infty}^{0} \frac{x}{(1+x^2)^2} dx + \int_{0}^{+\infty} \frac{x}{(1+x^2)^2} dx$

$v = 1 + x^2$
$dv = 2x\,dx$

$\displaystyle = \lim_{u \to -\infty} \left(\frac{1}{2} \int_u^0 \frac{2x}{(1+x^2)^2} dx \right) + \lim_{t \to +\infty} \left(\int_0^t \frac{x}{(1+x^2)^2} dx \right)$

$\displaystyle = \lim_{u \to -\infty} \left(-\frac{1}{2} \left[\frac{1}{1+x^2} \right]_u^0 \right) + \lim_{t \to +\infty} \left[-\frac{1}{2} \frac{1}{1+x^2} \right]_0^t$

$= -\frac{1}{2}[1-0] + \left(-\frac{1}{2}\right)[0-1]$

$= -\frac{1}{2} + \frac{1}{2} = 0$

(12) $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}} = \lim_{t \to 1^+} \left(\int_t^e \frac{dx}{x\sqrt{\ln x}} \right)$

$\displaystyle = \lim_{t \to 1^+} \int_{\ln t}^{\ln e} \frac{dv}{\sqrt{v}}$ $\quad v = \ln x$, $dv = \frac{dx}{x}$

$\displaystyle = \lim_{t \to 1^+} \left[2\sqrt{v} \right]_{\ln t}^1$

$\displaystyle = 2 \lim_{t \to 1^+} \left[1 - \sqrt{\ln t} \right] = 2$

(16) $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/5}} = \int_0^1 \frac{dx}{(x-1)^{2/5}} + \int_1^2 \frac{dx}{(x-1)^{2/5}}$ because $\ln 1 = 0$

$\displaystyle = \lim_{t \to 1^-} \left(\int_0^t (x-1)^{-2/5} dx \right) + \lim_{u \to 1^+} \left(\int_u^2 (x-1)^{-2/5} dx \right)$

$\displaystyle = \lim_{t \to 1^-} \frac{5}{3} \left[(x-1)^{3/5} \right]_0^t + \lim_{u \to 1^+} \frac{5}{3} \left[(x-1)^{3/5} \right]_u^2$

$\displaystyle = \frac{5}{3} \lim_{t \to 1^-} \left[(t-1)^{3/5} - (-1)^{3/5} \right] + \frac{5}{3} \lim_{u \to 1^+} \left[1^{3/5} - (u-1)^{3/5} \right]$

$= \frac{5}{3}\left[0 - (-1)^{3/5} \right] + \frac{5}{3}\left[1^{3/5} \right]$

$= \frac{10}{3}$ ✓

(17) $\displaystyle\int_1^{+\infty} \frac{dx}{x^3} = \lim_{t \to +\infty} \left(\int_1^t \frac{dx}{x^3} \right)$

$\displaystyle = \lim_{t \to +\infty} \left(\int_1^t x^{-3} dx \right)$

$\displaystyle = \lim_{t \to +\infty} \left[\frac{x^{-2}}{-2} \right]_1^t$

$\displaystyle = -\frac{1}{2} \lim_{t \to +\infty} \left[\frac{1}{t^2} - 1 \right] = \frac{1}{2}$

(19) $\displaystyle\int_0^4 \frac{dx}{x^4} = \lim_{u \to 0^+} \left(\int_u^4 x^{-4} dx \right)$

$\displaystyle = \lim_{u \to 0^+} \left[\frac{x^{-3}}{-3} \right]_u^4$

$\displaystyle = -\frac{1}{3} \lim_{u \to 0^+} \left[4^{-3} - \frac{1}{u^3} \right] = +\infty$

(23) $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x+1}} = \lim_{t \to +\infty} \left(2 \int_0^t \frac{dx}{2\sqrt{x+1}} \right)$ diverges

$\displaystyle = \lim_{t \to +\infty} 2\left[\sqrt{x+1} \right]_0^t$

$\displaystyle = 2 \lim_{t \to +\infty} \left[\sqrt{t+1} - 1 \right] = +\infty$

(25) $\displaystyle\int_e^{\infty} \frac{dx}{x \ln x} = \lim_{t \to +\infty} \left(\int_e^t \frac{dx}{x \ln x} \right)$ diverges

$\displaystyle = \lim_{t \to +\infty} \left(\int_{\ln e}^{\ln t} \frac{dv}{v} \right)$ $\quad v = \ln x$, $dv = \frac{dx}{x}$

$\displaystyle = \lim_{t \to +\infty} \left[\ln v \right]_{\ln e}^{\ln t}$

$\displaystyle = \lim_{t \to +\infty} \left[\ln(\ln t) - \underbrace{\ln(\underbrace{\ln e}_{1})}_{0} \right] = +\infty$ diverges

Chapter 9 Solving system of linear equations

System of linear equations

- linear eq of n variables:

$$\boxed{a_1x_1 + a_2x_2 + \dots + a_nx_n = b}$$

where $x_1, x_2, \dots, x_n$ are variables and $a_1, \dots, a_n$ are coefficients.

expl ① $3x + 4y = 7$ is a linear equation

② $2x_1 - 3x_2 + 3x_3 = 5$ is a linear equation.

- **Linear system**: A linear system of $\boxed{m}$ linear equations and n unknowns is

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m,2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

where $x_1, \dots, x_n$ are variables and a's and b's are constants.

expl $\begin{cases} 2x + y = 4 \\ x - 3y = -5 \end{cases}$

* Solution linear system

The solution of linear system is a sequence of n numbers $s_1, s_2, \dots, s_n$ which satisfies the system when we substitute $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$.

Note:

- Inconsistent. The system of equation is inconsistent if it has no solution.
- Consistent: The system of equation is consistent if it has one or infinite solutions.

expl Solve $\begin{cases} 2x + y = 4 \\ x - 3y = -5 \end{cases}$

Ans Method by substitution

$\begin{cases} 2x + y = 4 \quad (Eq_1) \ ✓ \\ x - 3y = -5 \quad (Eq_2) \ ✓ \end{cases}$

From (eq_1) $y = 4 - 2x$ by substitution in (Eq_2)

$x - 3(4 - 2x) = -5$

$x - 12 + 6x = -5$

$7x = 7 \Rightarrow x = 1$

So $y = 4 - 2 = 2$ $\quad y = 2$

$S = \{(1, 2)\}$

Second Method Gauss

$-2 \quad 6 \quad 10$

$$\left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & -3 & -5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} \boxed{1} & -3 & -5 \\ 2 & 1 & 4 \end{array}\right)$$

augmented matrix

$$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 7 & 14 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$0 \quad 3 \quad 6$

So $x = 1$ and $y = 2$

ex 1 page 443

$\begin{cases} x - y = 1 \\ x - 2y = -2 \end{cases}$

Augmented matrix $\begin{array}{c} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -2 & -2 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \end{array}$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & -3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - R_1 \end{array} \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right) \rightarrow \begin{array}{l} x = 4 \\ y = 3 \end{array}$$

Check $4 - 3 = 1$ ✓
$4 - 6 = -2$ ✓

ex 4 page 443

$\begin{cases} 2x + y = 1/3 \\ 6x + 3y = 1 \end{cases}$

Augmented matrix $\left(\begin{array}{cc|c} 2 & 1 & 1/3 \\ 6 & 3 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 1/2 & 1/6 \\ 1 & 1/2 & 1/6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 1/2 & 1/6 \\ 0 & 0 & 0 \end{array}\right) \leftarrow \text{infinitely solution}$$

$x + \frac{1}{2}y = 1/6$

$6x + 3y = 1$

$y = \frac{4}{3} - \frac{6x}{3}$

⚠ $S = \{(x, \frac{1}{3} - 2x); x \in \mathbb{R}\}$

- $\left(\begin{array}{cccc|c} 1 & \times & & \times & \times \\ 0 & \ddots & & \times & \times \\ 0 & 0 & 0 & 1 & \times \end{array}\right) \longrightarrow$ unique solution

- $\left(\begin{array}{cccc|c} 1 & \times & & & \times \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & \ddots & 1 & \\ 0 & \dots & \dots & 0 & \boxed{0} \end{array}\right) \longrightarrow$ Infinitely solution

- $\left(\begin{array}{ccccc|c} 1 & \cdot & \times & \cdot & \times & \times \\ 0 & \ddots & & & \times & \times \\ 0 & 0 & \ddots & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & X \neq 0 \end{array}\right) \longrightarrow$ No Solution

ex 9 pag 443

$\begin{cases} 2x - y = 3 \\ x - 3y = 7 \end{cases}$

Augmented matrix $\left(\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & -3 & 7 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array} \rightarrow$

$-2 \quad 6 \quad -14$

$$\left(\begin{array}{cc|c} \boxed{1} & -3 & 7 \\ 2 & -1 & 3 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \end{array} \rightarrow \left(\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & 5 & -11 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & 1 & -11/5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -\frac{11}{5} \end{array}\right)$$

Unique solution $x = 2/5$ & $y = \frac{-11}{5}$

Check $\begin{cases} 2x - y = 4/5 + \frac{11}{5} = \frac{15}{5} = 3 ✓ \\ x - 3y = \frac{2}{5} + \frac{33}{5} = \frac{35}{5} = 7 ✓ \end{cases}$

ex 22 page 443

$$\begin{cases} x+4y-3z=-13 \\ 2x-3y+5z=18 \\ 3x+y-2z=1 \end{cases}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & -3 & -13 \\ 2 & -3 & 5 & 18 \\ 3 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & -3 & -13 \\ 0 & -11 & 11 & 44 \\ 0 & -11 & 7 & 40 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & -3 & -13 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right) \qquad 0 \quad 11 \quad -11 \quad -44$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & -3 & -13 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ unique solution

$0 \quad 0 \quad 3 \quad 3$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & 0 & -10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$0 \quad -4 \quad 0 \quad 12$

So $x=2$; $y=-3$; $z=1$

check:

$$\begin{cases} x+4y-3z = 2+(-12)-3 = -13 \checkmark \\ 2x-3y+5z = 4+9+5 = 18 \checkmark \\ 3x+y-2z = 6-3-2 = 1 \checkmark \end{cases}$$

ex 25

$$\begin{cases} -x-2y+3z=-9 \\ 2x+y-z=5 \\ 4x-3y+5z=-9 \end{cases}$$

Augmented matrix

$-2 \quad -4 \quad 6 \quad -18$

$-4 \quad -8 \quad 12 \quad -36$

$$\left(\begin{array}{ccc|c} -1 & -2 & 3 & -9 \\ 2 & 1 & -1 & 5 \\ 4 & -3 & 5 & -9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & 1 & -1 & 5 \\ 4 & -3 & 5 & -9 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -3 & 5 & -13 \\ 0 & -11 & 17 & -45 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{5}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{17}{11} & \frac{45}{11} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{5}{3} & \frac{13}{3} \\ 0 & 0 & \frac{4}{33} & \frac{45}{11}-\frac{13}{3} \end{array}\right) \qquad -\frac{17}{11}+\frac{5}{3}=\frac{55-51}{33}$$

has unique solution.

ex 26

$$\begin{cases} 3x-2y+z=4 \\ 4x+y-2z=-12 \\ 2x-3y+z=7 \end{cases}$$

Augmented matrix

$-4 \quad -4 \quad 0 \quad 12$

$-2 \quad -2 \quad 0 \quad 6$

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 4 & 1 & -2 & -12 \\ 2 & -3 & 1 & 7 \end{array}\right)\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 4 & 1 & -2 & -12 \\ 2 & -3 & 1 & 7 \end{array}\right) R_1-R_3$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & -3 & -2 & 0 \\ 0 & -5 & 1 & 13 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 2/3 & 0 \\ 0 & -1 & 1/5 & \frac{13}{5} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & \frac{13}{15} & 13/5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad 0 \quad 0 \quad -2/3 \quad -2$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

unique solution

$$\begin{cases} x=-1 \\ y=-2 \\ z=3 \end{cases}$$

check:

$$\begin{cases} 3x-2y+z=-3+4+3=4 \checkmark \\ 4x+y-2z=-4-2-6=-12 \checkmark \\ 2x-3y+z=-2+6+3=7 \checkmark \end{cases}$$

Definitions

① **Matrix** A set of objects arranged in a rectangular array having in "m" rows and "n" columns. These objects can be numbers or functions are called an $m \times n$ matrix

expl ①

$$\begin{array}{c|c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{pmatrix} \end{array}$$ 2×3 matrix

② If $m=n$, we say a squared matrix

expl $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ 2×2 ; $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ -1 & 2 & 0 \end{pmatrix}$ 3×3

⚠

$$\begin{cases} a_{11}x_1+\dots+a_{1n}x_n=b_1 \\ \vdots \\ a_{m1}x_1+\dots+a_{mn}x_n=b_m \end{cases}$$

can be written as $\boxed{AX=B}$

with $A=\begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}$; $X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$B=\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

expl

$$\begin{cases} 2x+y-3z=4 \\ x-y+2z=3 \\ 3x+6y+z=5 \end{cases} \iff AX=B$$

$$\underbrace{\begin{pmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 6 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}}_{B}$$

The augmented matrix is

$$(A|B) = \left(\begin{array}{ccc|c} 2 & 1 & -3 & 4 \\ 1 & -1 & 2 & 3 \\ 3 & 6 & 1 & 5 \end{array}\right)$$

Operations on Matrices

- **Addition** Let $A=(a_{ij})$ and $B=(b_{ij})$ be 2 matrices of the same size then we define $A+B=(a_{ij}+b_{ij})$

expl Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}$

Then $A+B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 5 & 4 & -4 \end{pmatrix}$

$A-B = \begin{pmatrix} 1 & 1 & 4 \\ -2 & -4 & 6 \\ 5 & 0 & -2 \end{pmatrix}$

- **Multiplication by scalar**

Let $A = (a_{ij})$ be a matrix and k be a scalar then $(k \cdot A) = (k a_{ij})$ is a matrix

expl $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \end{pmatrix}$

then $2A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 6 & 10 \end{pmatrix}$;

$-A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & -3 & -5 \end{pmatrix}$

Note that

$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ zero matrix

is the identity for addition or substraction

expl $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

- **Multiplication of 2 matrices**

Def Let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le k}}$ $m \times k$ matrix

and $B = (b_{ij})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ $k \times n$ matrix

We can define the product of 2 matrices (number of columns of 1st matrix = the number of Rows of the second matrix)

$C = AB = (c_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

Where $c_{ij} = (a_{i1} \; a_{i2} \cdots a_{ik}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \end{pmatrix}$

$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj}$

expl Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 0 \end{pmatrix}$ 2×3 matrix

$B = \begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 5 \end{pmatrix}$ 3×2 matrix

So $AB = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ -7 & -8 \end{pmatrix}$

$1 \times 2 + 2 \times -1 + 3 \times 3$

$BA = \begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 16 & 6 \\ -1 & -2 & -3 \\ -7 & 21 & 9 \end{pmatrix}$

$AB \neq BA$

⚠ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of the multiplication

$A \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

ex1 page 445

Find $A + 2B - 3C$ if

$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$; $B \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}$; $C \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Ans

$A + 2B - 3C = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & -6 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -9 \end{pmatrix}$

$= \begin{pmatrix} -1 & 5 \\ -1 & -15 \end{pmatrix}$

ex2 Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$ Find B so that $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Ans

As $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

then $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - A$

So $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$

Def (Transposition of a matrix)

Suppose that $A = (a_{ij})$ is $m \times n$ matrix Then the transpose of A, denoted by A^T is an $n \times m$ matrix with entries $a_{ij}^T = a_{ji}$

ex3 Transpose of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

is $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

expl

if $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$

Compute AB & BA

Ans $AB = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$

⚠ $AB \neq BA$

⚠ $(A+B)C = AC + BC$

$A(B+C) = AB + AC$

$(AB)C = A(BC)$

$A \cdot O = O$

$A^2 = A \cdot A$

$A^3 = A^2 A = A \cdot A \cdot A$

expl If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; Find A^3

Ans $A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

So $A^3 = A^2 \cdot A = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$

Inverse of a matrix

Def Suppose that $A=(a_{ij})$ is an n-squared matrix. If there exists an n-squared matrix B such that $AB = BA = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ then B is called the <u>inverse matrix</u> of A denoted by A^{-1}

⚠ If A has an inverse we say that A is <u>invertible</u>. Otherwise is said singular.

The inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>exple</u> The inverse of $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

then $|A| = \det A = 2\times 3 - 5\times 1 = 1 \neq 0$

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

<u>check</u> $AA^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

⚠ (*) $AX = B$ if A is invertible then (*) has <u>unique</u> solution

$$\boxed{X = A^{-1}B}$$

<u>Def</u> (determinant of a matrix)

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det A = |A| = ad - bc$
- If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

then $\det A = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$

<u>Thm</u>: If $\det A \neq 0$ then A has an inverse.

Computing Inverse Matrices

find A^{-1} if $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

<u>1st method</u> $|A| = \det A = 2\cdot 3 - 1\cdot 5 = 1 \neq 0$

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

<u>2nd method</u>

$$(A|I) = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right)$$

$-2 \quad -6 \quad 0 \quad -2$

$$\to \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$0 \quad -3 \quad 3 \quad -6$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right) = (I|A^{-1})$$

So $A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

<u>exple</u> Find A^{-1} if $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

<u>Ans</u>: $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{matrix} 1 & -1 \\ 2 & -1 \\ -1 & 1 \end{matrix}$

$\det A = (1+1-2) - (-1+1+2)$
$= -2 \neq 0$ so A is invertible

$A^{-1} = ?$

$-2 \quad 2 \quad 2 \quad -2 \quad 0 \quad 0$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right)$$

$0 \quad 0 \quad -3 \quad 3/2 \quad 0 \quad 3/2$

$$\to \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1 & 3/2 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1 & 3/2 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1/2 & 1 & 3/2 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 1 & 3/2 \\ -1/2 & 0 & -1/2 \end{pmatrix}$

<u>Check</u>:

$$AA^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 1 & 3/2 \\ -1/2 & 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

✓

<u>ex</u> 1, 3, 8, 10, 11, 14, 15, 21, 22, 23, 24, 27, 32, 40, 42 page 465.

ex 3 Determine D so that $A+B = 2A-B+D$

Ans $D = A+B-2A+B$
$= 2B - A$
$= 2\begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ 4 & 11 \end{pmatrix}$

ex 8: $3C - B + \frac{1}{2}A =$

$$3\begin{pmatrix} -2 & 0 & 4 \\ 1 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 1 & -3 & -3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{21}{2} & \frac{5}{2} & \frac{15}{2} \\ 2 & -7 & \frac{5}{2} \\ -1 & 2 & 10 \end{pmatrix}$$

ex 10 Determine D so that $A + 4B = 2(A+B) + D$

Ans $D = A + 4B - 2(A+B)$
$= A + 4B - 2A - 2B$
$= 2B - A$
$= 2\begin{pmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 1 & -3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix}$
$= \begin{pmatrix} 9 & -5 & 9 \\ 2 & -4 & 1 \\ 2 & -4 & -8 \end{pmatrix}$

ex 11 Show that $A+B = B+A$

$A + B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 1 & -3 & -3 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 \\ 4 & 4 & 2 \\ 1 & -5 & -1 \end{pmatrix}$

$B + A = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 0 & 1 \\ 1 & -3 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 \\ 4 & 4 & 2 \\ 1 & -5 & -1 \end{pmatrix}$

So $A+B = B+A$

ex 14 The transpose of $A = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ is

$A^T = \begin{pmatrix} 2 & -3 & 5 \end{pmatrix}$

ex 15
The transpose of $A = \begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & -4 \end{pmatrix}$

is $A^T = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 3 & -4 \end{pmatrix}$

ex 21 Let $A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$; $B\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$; $C\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

(i) $AB = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 0 & 5 \end{pmatrix}$

$BA = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}$

⚠ $AB \neq BA$

(ii) $ABC = (AB)C = \begin{pmatrix} -2 & -3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$
$= \begin{pmatrix} -2 & -1 \\ 0 & -5 \end{pmatrix}$

$AC = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$

$CA = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & -2 \end{pmatrix}$

So $AC \neq CA$

ex 33 Let $A = \begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix}$
Find A^2; A^3; A^4

Ans $A^2 = A \cdot A = \begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 8 \end{pmatrix}$

$A^3 = A^2 \cdot A = \begin{pmatrix} 3 & -1 \\ -1 & 8 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ -10 & -25 \end{pmatrix}$

$A^4 = A^3 \cdot A = \begin{pmatrix} 7 & 6 \\ -10 & -25 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 8 & -11 \\ 5 & 65 \end{pmatrix}$

ex 43 Show that the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
is $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

Ans As $A \cdot B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$
So $B = A^{-1}$

ex 45 Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$
$\det A = -3 - 2 = -5 \neq 0$ So A has an inverse

$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 3 & -1 \\ -2 & -1 \end{pmatrix}$

$A^{-1} = -\frac{1}{5}\begin{pmatrix} 3 & -1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -3/5 & 1/5 \\ 2/5 & 1/5 \end{pmatrix}$

Check $AA^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3/5 & 1/5 \\ 2/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

⚠ $A\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\det A = ad - bc \neq 0$
then A is invertible and $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Inverse of $B = \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}$ is

$B^{-1} = \frac{1}{4-(-6)}\begin{pmatrix} 2 & 2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 \\ -3/10 & 1/5 \end{pmatrix}$

ex 47 Show that $(AB)^{-1} = B^{-1}A^{-1}$

$A\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$; $A^{-1} = \begin{pmatrix} -3/5 & 1/5 \\ 2/5 & 1/5 \end{pmatrix}$

$B\begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}$; $B^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ -3/10 & 1/5 \end{pmatrix}$

Then
$AB = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 13 & 2 \end{pmatrix}$

$(AB)^{-1} = \frac{1}{2-52}\begin{pmatrix} 2 & -4 \\ -13 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{25} & \frac{2}{25} \\ \frac{13}{50} & -\frac{1}{50} \end{pmatrix}$

$B^{-1}A^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ -3/10 & 1/5 \end{pmatrix}\begin{pmatrix} -3/5 & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} -1/25 & \frac{2}{25} \\ \frac{13}{50} & -\frac{1}{50} \end{pmatrix}$

so $(AB)^{-1} = B^{-1}A^{-1}$

ex 55 Use the determinant to determine the matrix whether $A = \begin{pmatrix} 4 & -1 \\ 8 & -2 \end{pmatrix}$ is invertible
Ans $\det A = -8-(-8) = 0$ so A is singular.

ex 60 Find A^{-1} if $A=\begin{pmatrix}1&2\\0&3\end{pmatrix}$

Ans $\det A = 3 \neq 0$ so A is invertible

$\triangle$ $\det\begin{pmatrix}a&\alpha\\0&b\\0&0&c\end{pmatrix}=abc$

So $A^{-1}=\frac{1}{3}\begin{pmatrix}3&-2\\0&1\end{pmatrix}=\begin{pmatrix}1&-2/3\\0&1/3\end{pmatrix}$

ex 65 $C=\begin{pmatrix}1&3\\1&3\end{pmatrix}$ is not invertible (singular)

$\det C = 0$

ex 67 Find A^{-1} for $A=\begin{pmatrix}2&-1&-1\\2&1&1\\-1&1&-1\end{pmatrix}\begin{matrix}2&-1\\2&1\\-1&1\end{matrix}$

Ans $\det A=(-2+1-2)-(1+2+2)$
$= -3-5=-8\neq 0$
So A is invertible.

$$(A/I)=\left(\begin{array}{ccc|ccc}2&-1&-1&1&0&0\\2&1&1&0&1&0\\-1&1&-1&0&0&1\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|ccc}-1&1&-1&0&0&1\\-2&-1&-1&1&0&0\\2&1&1&0&1&0\end{array}\right)$$

$$\begin{array}{cccccc}-2&2&-2&0&0&2\end{array}$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&-1&1&0&0&-1\\0&1&-3&1&0&2\\0&3&-1&0&1&2\end{array}\right)$$

$$\begin{array}{cccccc}0&-3&9&-3&0&-6\end{array}$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&-1&1&0&0&-1\\0&1&-3&1&0&2\\0&0&8&-3&1&-4\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&-1&1&0&0&-1\\0&1&-3&1&0&2\\ \hline 0&0&1&-3/8&1/8&-1/2\end{array}\right)$$

$$\begin{array}{cccccc}0&0&3&-9/8&3/8&-3/2\end{array}$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&-1&1&0&0&-1\\0&1&0&-1/8&3/8&1/2\\0&0&1&-3/8&1/8&-1/2\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&-1&0&3/8&-1/8&-1/2\\0&1&0&-1/8&3/8&1/2\\0&0&1&-3/8&1/8&-1/2\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&1/4&1/4&0\\0&1&0&-1/8&3/8&1/2\\0&0&1&-3/8&1/8&-1/2\end{array}\right)$$

So $A^{-1}=\begin{pmatrix}1/4&1/4&0\\-1/8&3/8&1/2\\-3/8&1/8&-1/2\end{pmatrix}$

Check.

$AA^{-1}=\begin{pmatrix}2&-1&-1\\2&1&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}1/4&1/4&0\\-1/8&3/8&1/2\\-3/8&1/8&-1/2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

ex 70 page 466

Find A^{-1} if $A=\begin{pmatrix}-1&0&2\\-1&-2&3\\0&2&-1\end{pmatrix}\begin{matrix}-1&0\\-1&-2\\0&2\end{matrix}$

Ans $\det A=(-2+0-4)-(0-6+0)$
$= -6+6=0$
So A is not invertible.

$$(A|I)\rightarrow\left(\begin{array}{ccc|ccc}-1&0&2&1&0&0\\-1&-2&3&0&1&0\\0&2&-1&0&0&1\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&0&-2&-1&0&0\\0&-2&1&-1&1&0\\0&2&-1&0&0&1\end{array}\right)$$

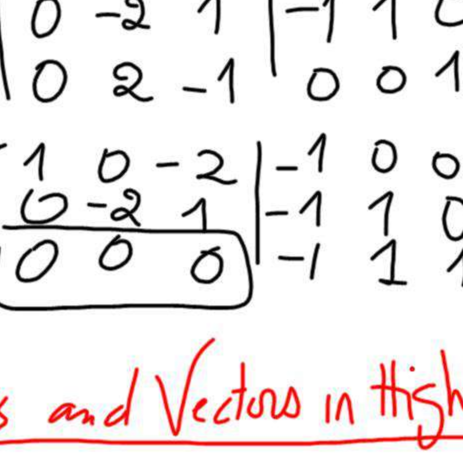
$$\rightarrow\left(\begin{array}{ccc|ccc}1&0&-2&-1&0&0\\0&-2&1&-1&1&0\\ \hline 0&0&0&-1&1&1\end{array}\right)$$

$\times$

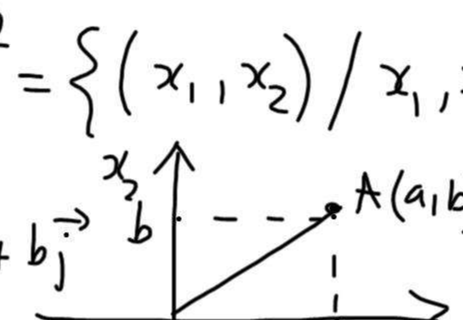
9.4.1 Points and Vectors in Higher dimensions

- The plane $\mathbb{R}^2=\{(x_1,x_2)\,/\,x_1,x_2\in\mathbb{R}\}$

$\vec{OA}=a\vec{i}+b\vec{j}$
$=(a,b)$
(i,j)



- The space $\mathbb{R}^3=\{(x,y,z)\,/\,x\in\mathbb{R},\ y\in\mathbb{R},\ z\in\mathbb{R}\}$

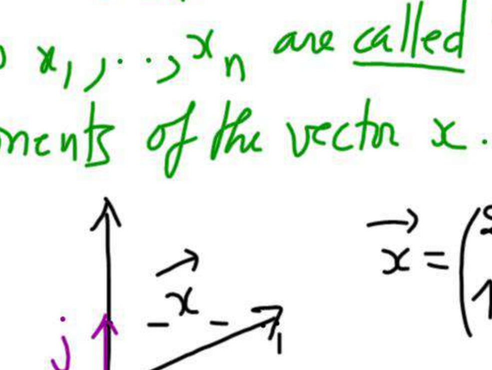


- $\mathbb{R}^n=\{(x_1,\dots,x_n)\,/\,x_i\in\mathbb{R}\}$

Def A vector in n-dimensional space is an ordered n-tuple

$X=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=x_1\vec{e}_1+x_2\vec{e}_2+\dots+x_n\vec{e}_n$

The numbers $x_1,\dots,x_n$ are called the components of the vector x.



$\vec{x}=\begin{pmatrix}2\\1\end{pmatrix}=2\vec{i}+\vec{j}$

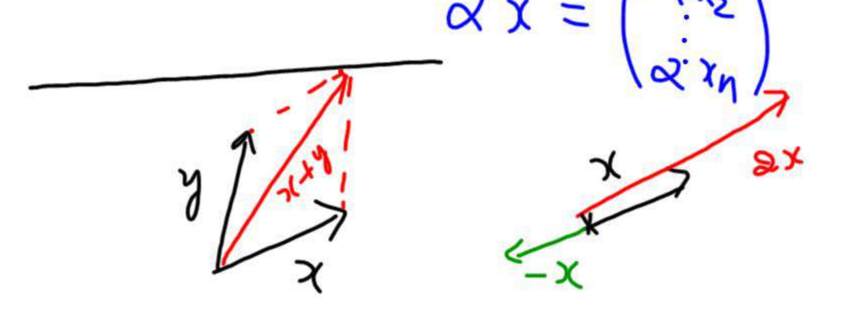
Properties

① Let $x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\in\mathbb{R}^n$ & $y\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\in\mathbb{R}^n$

Then $x+y=\begin{pmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{pmatrix}$

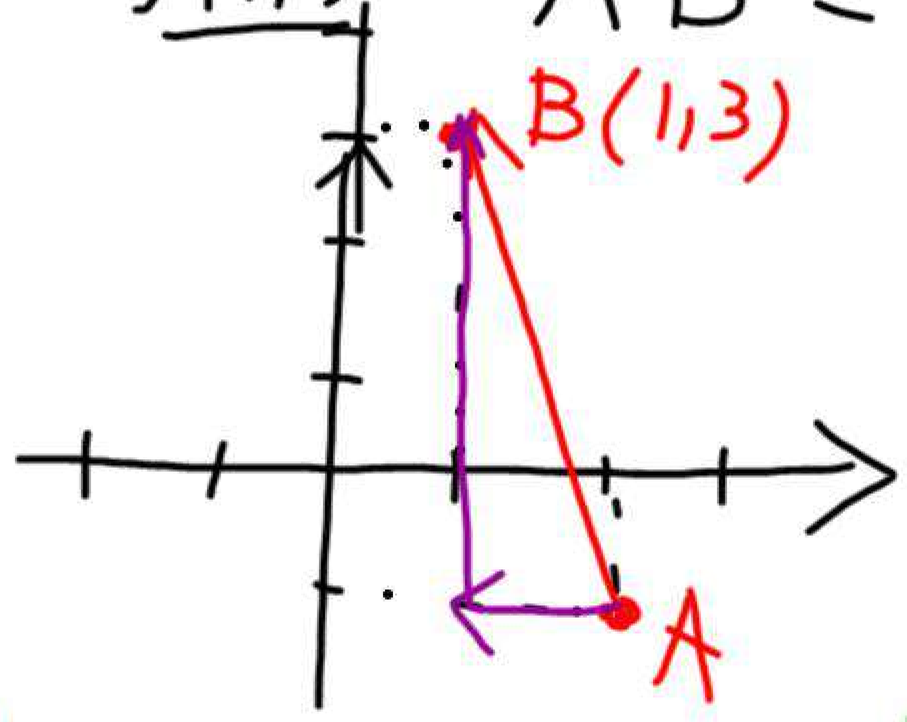
② Let $\alpha\in\mathbb{R}$ then $\alpha.x$ is also a vector

$\alpha x=\begin{pmatrix}\alpha x_1\\\alpha x_2\\\vdots\\\alpha x_n\end{pmatrix}$



expl · Find the vector representation of $\overrightarrow{AB}$ if $A(2,-1)$ & $B(1,3)$

Ans $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix} = \begin{pmatrix} 1-2 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$



Def (Norm of a vector)

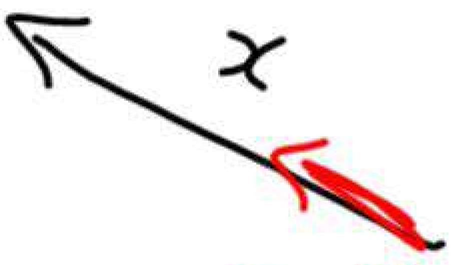
Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a vector, the norm of x is denoted $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

expl if $X = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ then $\|X\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$

Def (Unit vector)

Let X be a vector then $\frac{X}{\|X\|}$ is a unit vector

expl if $X = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ then the unit vector of x is $\frac{1}{\|X\|} x = \frac{1}{\sqrt{17}} x = \begin{pmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix}$



$v = \frac{x}{\|x\|}$: unit vector $(\|v\| = 1)$

Def (Dot product)

Let $X \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ & $Y \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ be 2 vectors in $\mathbb{R}^n$, we define the dot product

$$\langle x | y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

⚠ $$\boxed{x \cdot y = \|x\| \, \|y\| \cos(\widehat{x, y})}$$

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

Chapter 10 Multivariable Calculus

§ Functions of 2 or more independent variables

Def. Suppose $D \subset \mathbb{R}^n$. Then a real valued function f on D assigns a real number to each element in D and we write

$$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n)$$

The set D is the <u>domain</u> of the function f and the set $\left\{ w \in \mathbb{R} \,/\, f(x_1, \dots, x_n) = w \text{ for some } (x_1, \dots, x_n) \in D \right\}$ is the range of the function f.

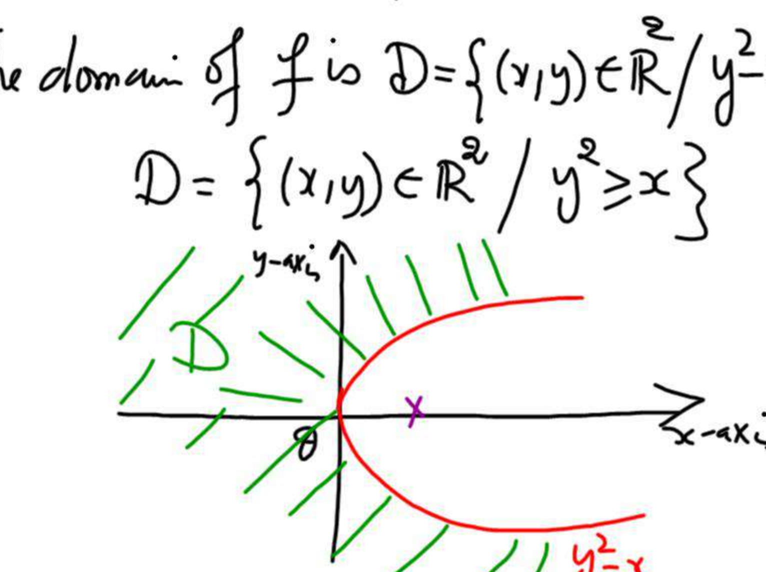
expl Let $f(x, y, z) = \dfrac{xy}{z^2}$ is defined on $\mathbb{R}^3 \setminus \{ z = 0 \}$

$$f(2,3,-1) = \frac{2 \times 3}{(-1)^2} = 6$$
$$f(-1,2,3) = \frac{(-1) \times 2}{(3)^2} = -2/9$$

exple 2: Find the largest possible domain of the function $f(x,y) = \sqrt{y^2 - x}$

The domain of f is $D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, y^2 - x \geq 0 \right\}$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, y^2 \geq x \right\}$$



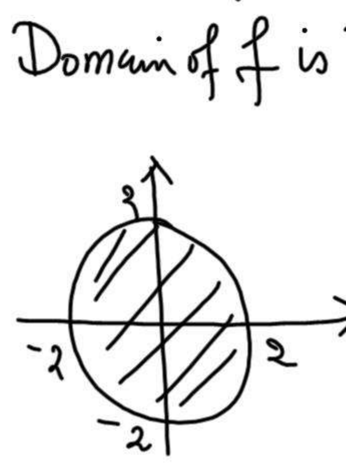
Def: If f is a function of 2 independent variables with domain D, then the graph of f is the set of all points (x, y, z) such that $z = f(x,y)$.

$$\text{Graph} f = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, z = f(x,y),\ (x,y) \in D \right\}$$
(Surface)

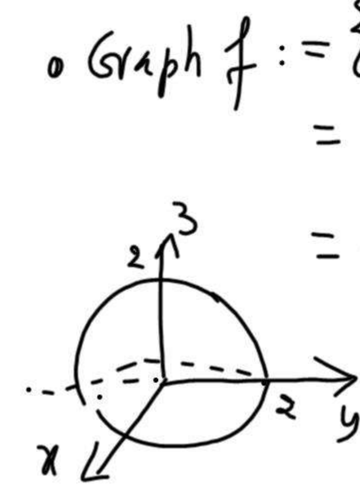
⚠ Level curves $= \left\{ (x,y) \,/\, f(x,y) = c \right\}$

exple: $z = f(x,y) = \sqrt{4 - x^2 - y^2}$

- Domain of f is $D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, 4 - x^2 - y^2 \geq 0 \right\}$
  $= \left\{ (x,y) \in \mathbb{R}^2 \,/\, x^2 + y^2 \leq 2^2 \right\}$
  $=$ Disk centered at the origin and radius $R = 2$



- Graph $f := \left\{ (x,y,z) \,/\, z = f(x,y) = \sqrt{4 - x^2 - y^2} \right\}$
  $= \left\{ (x,y,z) \,/\, z^2 = 4 - x^2 - y^2,\ z \geq 0 \right\}$
  $= \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 4 \right\}$
  $=$ half sphere $z \geq 0$



§ Partial derivatives

Def: Suppose that f is a function of 2 independent variables x & y. We define the partial derivative of f with respect to x as

- $f_x(x,y) = \dfrac{\partial f}{\partial x}(x,y) := \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x,y)}{h}$

Also the partial derivative of f with respect to y as

- $f_y(x,y) = \dfrac{\partial f}{\partial y}(x,y) = \lim\limits_{h \to 0} \dfrac{f(x, y+h) - f(x,y)}{h}$

exple Using the definition of Partial derivatives find $\dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$ if
$$f(x,y) = 2xy^2 + 3x^2 - y + 1$$

<u>Ans</u> As f is a polynomial function of 2 variables then f is defined, continuous and differentiable on $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$
$$= \lim_{h \to 0} \frac{\left(2(x+h)y^2 + 3(x+h)^2 - y + 1\right) - \left(2xy^2 + 3x^2 - y + 1\right)}{h}$$
$$= \lim_{h \to 0} \frac{2xy^2 + 2hy^2 + 3x^2 + 6xh + 3h^2 - y + 1 - 2xy^2 - 3x^2 + y - 1}{h}$$
$$= \lim_{h \to 0} \left(2y^2 + 6x + 3h\right) = 2y^2 + 6x$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$
$$= \lim_{h \to 0} \frac{\left[2x(y+h)^2 + 3x^2 - (y+h) + 1\right] - \left[2xy^2 + 3x^2 - y + 1\right]}{h}$$
$$= \lim_{h \to 0} \frac{2xy^2 + 4xyh + 2xh^2 + 3x^2 - y - h + 1 - 2xy^2 - 3x^2 + y - 1}{h}$$
$$= \lim_{h \to 0} \left(4xy + 2xh - 1\right) = 4xy - 1$$

exple Find $\dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$ for
$$f(x,y) = y e^{xy}$$

<u>Ans</u>:
- $\dfrac{\partial f}{\partial x}(x,y) = y \dfrac{\partial}{\partial x}(xy)\, e^{xy}$
  $= y^2 e^{xy}$

- $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{\partial}{\partial y}(y) \cdot e^{xy} + y \dfrac{\partial}{\partial y}\left(e^{xy}\right)$
  $= e^{xy} + xy e^{xy}$
  $= (1 + xy) e^{xy}$

Exple Find f_x & f_y for $f(x,y) = \dfrac{\sin(xy)}{x^2+\cos y}$

Ans:

$$f_x(x,y) = \frac{\partial}{\partial x}\left[\frac{\sin(xy)}{x^2+\cos y}\right]$$

$$= \frac{y\cos(xy)(x^2+\cos y) - \sin(xy)(2x)}{(x^2+\cos y)^2}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{\sin(xy)}{x^2+\cos y}\right)$$

$$= \frac{(x\cos(xy))(x^2+\cos y) - \sin(xy)(-\sin y)}{(x^2+\cos y)^2}$$