Chapter 7 (adopted to be CH6) Eigenvalues and Eigenvectors

7.1 Eigenvalues, Eigenvectors & Eigenspaces

7.2 Diagonalization

Introduction to Eigenvalue Problem

- Eigenvalue problem:

  If A is an $n\times n$ matrix, do there exist $n\times 1$ nonzero matrices $\boldsymbol{x}$ such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- Eigenvalue and eigenvector:

  A : an $n\times n$ matrix

  λ : a scalar

  $\boldsymbol{x}$: a $n\times 1$ nonzero column matrix

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

(The fundamental equation for the *Eigenvalue problem*)

- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix} \qquad x_1=\begin{bmatrix}1\\ 1\end{bmatrix} \qquad x_2=\begin{bmatrix}2\\ -1\end{bmatrix}$$

Eigenvalue

$$Ax_1=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}5\\ 5\end{bmatrix}=5\begin{bmatrix}1\\ 1\end{bmatrix}=5x_1$$

Eigenvector

Eigenvalue

$$Ax_2=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}-2\\ 1\end{bmatrix}=-1\begin{bmatrix}2\\ -1\end{bmatrix}=(-1)x_2$$

Eigenvector

- Question:

  Given an $n \times n$ matrix A, how can you find the eigenvalues and corresponding eigenvectors?

- Note:

  $$Ax = \lambda x \;\Rightarrow\; (\lambda I - A)x = 0$$ (homogeneous system)

  If $(\lambda I - A)x = 0$ has nonzero solutions if $\det(\lambda I - A) = 0$

- **Characteristic equation** of $A \in M_{n\times n}$:

  $$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

- Ex 2: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Sol: ***Characteristic equation***:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 5, -1$$

Eigenvalues: $\lambda_1 = 5, \lambda_2 = -1$

Eigenvectors:

$$(1)\lambda_1 = 5 \qquad \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Gives $x_1 = x_2$ **then** $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \neq 0$

$$(2)\lambda_2 = -1 \qquad \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Gives $x_1 = -2x_2$ **then** $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ t \neq 0$

- Ex 3: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Sol: Characteristic equation:

You will find

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -1 & \lambda - 2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1)(\lambda - 3) = 0$$

The eigenvalues: $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 3$

means reduced to:
(since syst is homogeneous $(\lambda_1 I - A)x = 0$)

$$\lambda_1 = 1$$

$$\Rightarrow \lambda_1 I - A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

No constraints on x_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors}: \quad t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = -1$$

$$\Rightarrow \lambda_2 I - A = \begin{bmatrix} -2 & -2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors}: \quad t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 3$$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & -2 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors:} \quad t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

7.1 Reminder and Eigenspaces

- Eigenvalue problem:

If A is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- Eigenvalue and eigenvector:

A : an $n \times n$ matrix

λ : a scalar

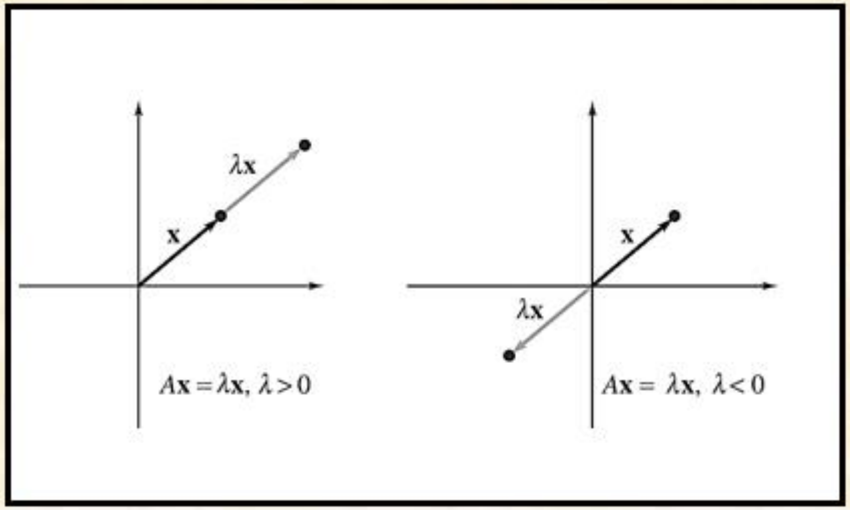
$\boldsymbol{x}$: a nonzero vector in R^n

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

- Geometrical Interpretation



consider a geometric interpretation in R^2 .
If λ is an eigenvalue of a matrix A and x is an eigenvector of A corresponding to λ, then multiplication of x by the matrix A produces a vector λx that is parallel to x, as shown in the figures.

- Ex 1: (Verifying eigenvalues and eigenvectors)

A- Verify that $A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$ has eigenvectors $x_1=\begin{bmatrix}1\\ 0\end{bmatrix}$ $x_2=\begin{bmatrix}0\\ 1\end{bmatrix}$

Eigenvalue

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=2\begin{bmatrix}1\\ 0\end{bmatrix}=2x_1$$

Eigenvector

Eigenvalue

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=-1\begin{bmatrix}0\\ 1\end{bmatrix}=(-1)x_2$$

Eigenvector

Ex 1: (Verifying eigenvalues and eigenvectors)

B- Verify that $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has eigenvectors $\mathbf{x}_1 = (-3, -1, 1)$ and $\mathbf{x}_2 = (1, 0, 0)$

Multiplying $\mathbf{x}_1$ on the left by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_1 = (-3, -1, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.
Similarly, multiplying $\mathbf{x}_2$ on the left by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (1, 0, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

- Thm 7.1: (The ***Eigenspace*** of ***A*** corresponding to λ)

If ***A*** is an *n*×*n* matrix with an eigenvalue $\boldsymbol{\lambda}$, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n : $\{\mathbf{x}: \mathbf{x} \text{ is an eigenvector of } \lambda\} \cup \{\mathbf{0}\}$

This subspace is called ***the Eigenspace of*** $\boldsymbol{\lambda}$.

Pf: x_1 and x_2 are eigenvectors corresponding to λ

$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$

$(1)\ A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$

$(i.e.\ \ x_1 + x_2$ is an eigenvector corresponding to $\lambda)$

$(2)\ A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

- Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\mathrm{v} = (x, y)$

$$A\mathrm{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

⇨ Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis.

For a vector on the *x*-axis

Eigenvalue $\lambda_1 = -1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

Reminder:

$\det(\lambda I - A) = 0$ gives eigenvalues: -1 and 1

Then for each eigenvalue we get eigenvector using

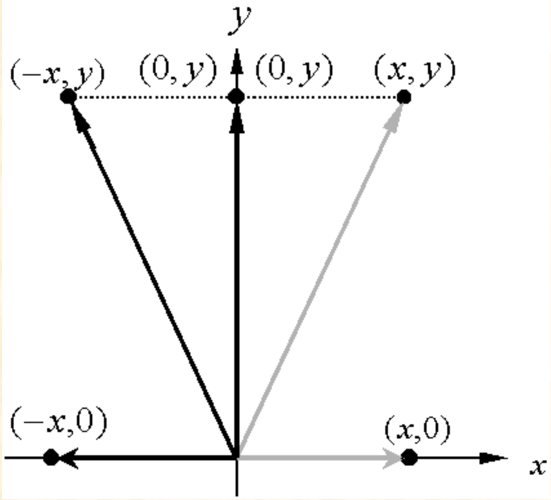
$$(\lambda_1 I - A)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a vector on the *y*-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix *A* corresponds to a **reflection** in the *y*-axis.



The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

- Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

  Let A be an $n\times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)x = 0$.

- Note:

  $Ax = \lambda x \;\Rightarrow\; (\lambda I - A)x = 0$ (homogeneous system)

  If $(\lambda I - A)x = 0$ has nonzero solutions if $\det(\lambda I - A) = 0$

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A:

$$\boxed{\det(\lambda I - A) = 0}$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda \mathrm{I} - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

$$\text{Eigenvalues}: \ \lambda_1 = -1, \lambda_2 = -2$$

$$(1)\lambda_1 = -1 \Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad \text{then} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Equivalent to:
$$\begin{cases} x_1 - 4x_2 = 0 \\ 0x_1 + 0x_2 = 0 \quad \text{(no constraints)} \end{cases}$$

$$(2)\lambda_2 = -2 \Rightarrow (\lambda_2 I - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Check : $Ax = \lambda_i x$

Details on the solution:

So, the characteristic equation is $(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A. To find the corresponding eigenvectors, solve the homogeneous linear system represented by $(\lambda I - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I - A = \begin{bmatrix} -1-2 & 12 \\ -1 & -1+5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$$

which row reduces to $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$, showing that $x_1 - 4x_2 = 0$. Letting $x_2 = t$, you can conclude that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

For $\lambda_2 = -2$, you have

$$(-2)I - A = \begin{bmatrix} -2-2 & 12 \\ -1 & -2+5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

Letting $x_2 = t$, you can conclude that every eigenvector of λ_2 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

- Ex 5: (Finding eigenvalues and eigenvectors)

Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix}\lambda-2 & -1 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda-2\end{vmatrix} = (\lambda-2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda I - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Notes:

  (1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynominal, then λ_1 has multiplicity *k*.

  (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex 6： Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 1)^2 (\lambda - 2)(\lambda - 3) = 0$$

$$\text{Eigenvalues}: \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

$$(1)\ \lambda_1 = 1$$

$$\Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix},\ \ s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 I - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\lambda_3 = 3$$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \ t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Thm 7.3: (Eigenvalues of triangular matrices)

  If A is an $n\times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 7: (Finding eigenvalues for diagonal and triangular matrices)

$$(a)A=\begin{bmatrix}2&0&0\\-1&1&0\\5&3&-3\end{bmatrix}\quad (b)A=\begin{bmatrix}-1&0&0&0&0\\0&2&0&0&0\\0&0&0&0&0\\0&0&0&-4&0\\0&0&0&0&3\end{bmatrix}$$

Sol:

$$(a)\ \left|\lambda I-A\right|=\begin{vmatrix}\lambda-2&0&0\\1&\lambda-1&0\\-5&-3&\lambda+3\end{vmatrix}=(\lambda-2)(\lambda-1)(\lambda+3)$$

$$\lambda_1=2,\ \lambda_2=1,\ \lambda_3=-3$$

$$(b)\ \lambda_1=-1,\ \lambda_2=2,\ \lambda_3=0,\ \lambda_4=-4,\ \lambda_5=3$$

Keywords in Section 7.1:

- eigenvalue problem: مسألة القيمة الذاتية
- eigenvalue: قيمة ذاتية
- eigenvector: متجه ذاتي
- characteristic polynomial: متعددة الحدود المميزة
- characteristic equation: المعادلة المميزة
- eigenspace: فضاء ذاتي
- multiplicity: تعددية

7.2 Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $\boldsymbol{P^{-1}AP}$ is diagonal?

- Diagonalizable matrix:

  a square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

  (we say: P diagonalizes A)

- Notes:

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

  (2) The eigenvalue problem is **closely related** to the diagonalization problem.

- Thm 7.4: (Similar matrices have the same eigenvalues)

  If A and B are **similar** $n \times n$ matrices, then they have the **same eigenvalues**.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda I P - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}$$

A and B have the same characteristic polynomial.
Thus A and B have the same eigenvalues.

Consequence: ***diagonalizing a matrix facilitates finding its eigenvalues***

- Ex 1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{Eigenvalues}: \ \lambda_1=4, \lambda_2=-2, \lambda_3=-2$$

$$(1)\lambda=4 \Rightarrow \ \text{Eigenvector:} p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \ \text{(check)}$$

$$(2)\lambda = -2 \Rightarrow \text{Eigenvector}: p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{(check)}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Thm 7.5: (Condition for diagonalization)

  An ***n*×*n*** matrix *A* is **diagonalizable** if and only if it **has *n* linearly independent eigenvectors**.

Pf:

$(\Rightarrow) A$ is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$ and $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$PD = [p_1 \mid p_2 \mid \cdots \mid p_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n]$$

$$AP = A[\, p_1 \mid p_2 \mid \cdots \mid p_n \,] = [\, Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n \,]$$

$\because AP = PD$

$\therefore Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$

(*i.e.* the column vector p_i of P are eigenvectors of A)

$\because P$ is invertible $\Rightarrow p_1, p_2, \cdots, p_n$ are linearly independent.

$\therefore$ A has n linearly independent eigenvectors.

$(\Leftarrow)$ A has n linearly independent eigenvectors $p_1, p_2, \cdots p_n$

with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

i.e. $Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$

Let $P = [\, p_1 \mid p_2 \mid \cdots \mid p_n \,]$

$$AP = A[p_1 \mid p_2 \mid \cdots \mid p_n]$$
$$= [Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n]$$
$$= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n]$$
$$= [p_1 \mid p_2 \mid \cdots \mid p_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD$$

$\because p_1, p_1, \cdots, p_n$ are linearly independent $\Rightarrow P$ is invertible

$\therefore P^{-1}AP = D$

$\Rightarrow A$ is diagonalizable

Note: If ***n*** linearly independent vectors **do not exist**, then an ***n*×*n*** matrix *A* is **not diagonalizable**.

- Ex 4: (A matrix that is **not** diagonalizable)

Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

Eigenvalue : $\lambda_1=1$

$$\lambda I-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\Rightarrow \text{Eigenvector}:\ p_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

- Steps for **diagonalizing** an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $p_1, p_2, \cdots, p_n$ for A with their corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Step 2: Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } Ap_i = \lambda_i p_i, \; i = 1, 2, \ldots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex 5: (Diagonalizing a matrix)

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \ (\textit{diagonal form})$$

- Notes: k is a positive integer

$$(1)\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k=\begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D=P^{-1}AP$$

$$\begin{aligned} \Rightarrow D^k &= (P^{-1}AP)^k \\ &= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}AA\cdots AP \\ &= P^{-1}A^kP \end{aligned}$$

also $\therefore A^k = PD^kP^{-1}$

- Thm 7.6: (Sufficient conditions for diagonalization)

  If an $\boldsymbol{n \times n}$ matrix A has $\boldsymbol{n}$ distinct eigenvalues, then the corresponding eigenvectors are linearly independent and hence A is diagonalizable.

- Ex 7: (Determining whether a matrix is diagonalizable)

$$A=\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)

Keywords in Section 7.2:

- diagonalization problem: مسألة التقطير (الجدولة)
- diagonalization: تقطير (جدولة)
- diagonalizable matrix: مصفوفة قابلة للتقطير (للجدولة)