Chapter 4

Vector Spaces الفضاءات الاتجاهية

4.1 Vectors in R^n

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4.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $(x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space

= set of all real number

$n = 2$ R^2 = 2-space

= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ R^4 = 4-space

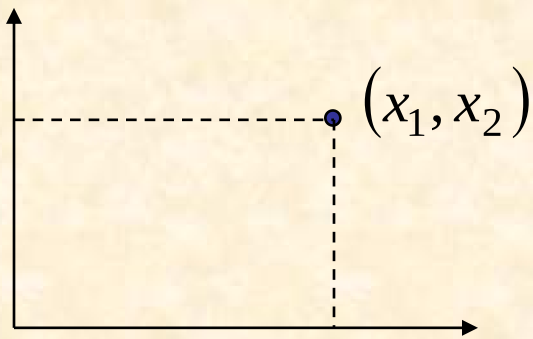
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

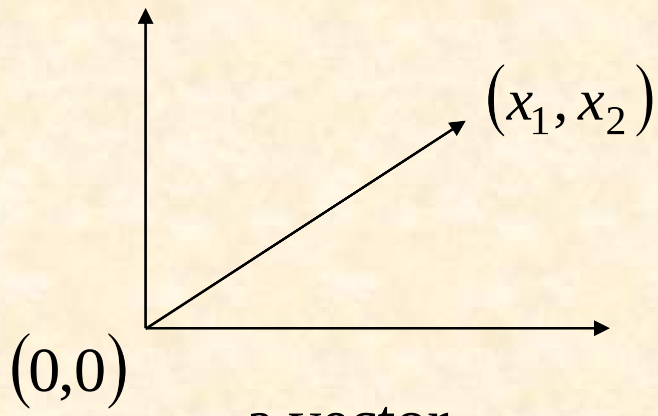
  (1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

  (2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

$\mathbf{u} = (u_1, u_2, \cdots, u_n),\ \mathbf{v} = (v_1, v_2, \cdots, v_n)$ (two vectors in R^n)

- Equal:

  $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2, \cdots, u_n = v_n$

- Vector addition (the sum of **u** and **v**):

  $$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

- Scalar multiplication (the scalar multiple of **u** by *c*):

  $$c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, ..., -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, ..., u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, ..., 0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(6) $c\mathbf{u}$ is a vector in R^n

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Ex 5: (Vector operations in R^4)

Let $\mathbf{u}=(2, -1, 5, 0)$, $\mathbf{v}=(4, 3, 1, -1)$, and $\mathbf{w}=(-6, 2, 0, 3)$ be vectors in R^4. Solve $\mathbf{x}$ for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)
$$\begin{aligned}\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\ &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\ &= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\ &= (4-4+18, -2-3-6, 10-1-0, 0+1-9) \\ &= (18, -11, 9, -8).\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\quad 3(\mathbf{x}+\mathbf{w}) &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}+3\mathbf{w} &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}-\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
\mathbf{x} &= \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w} \\
&= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right) \\
&= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)
\end{aligned}
$$

- Thm 4.3: (Properties of additive identity and additive inverse)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

The vector $\mathbf{x}$ is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_n$,

if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

Given $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0,1,4)$, $\mathbf{v} = (-1,1,2)$, and $\mathbf{w} = (3,1,2)$ in R^3, find a, b, and c such that $\mathbf{x} = a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$.

Sol:

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

a 1×n row matrix (row vector): $\mathbf{u}=[u_1,\ u_2,\cdots,u_n]$

or

a n×1 column matrix (column vector): $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1,u_2,\cdots,u_n]+[v_1,v_2,\cdots,v_n]$$
$$=[u_1+v_1,u_2+v_2,\cdots,u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

$$c\boldsymbol{u}=c[u_1,\ u_2,\cdots,\ u_n]$$
$$=[cu_1,cu_2,\cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Keywords in Section 4.1:

- ordered *n*-tuple : زوج نوني مرتب
- *n*-space : فضاء نوني
- equal : مساوي
- vector addition : جمع متجهي
- scalar multiplication : ضرب عددي
- negative : سالب
- difference : الفرق
- zero vector : متجه صفري
- additive identity : محايد جمعي
- additive inverse : معاكس جمعي

4.2 Vector Spaces

- Vector spaces:

  Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

  Addition:

  (1) **u**+**v** is in V

  (2) **u**+**v**=**v**+**u**

  (3) **u**+(**v**+**w**)=(**u**+**v**)+**w**

  (4) V has a zero vector **0** such that for every **u** in V, **u**+**0**=**u**

  (5) For every **u** in V, there is a vector in V denoted by –**u** such that **u**+(–**u**)=**0**

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V : nonempty set

c : scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}$: vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}$: scalar multiplication

$(V,\ \ +,\ \ \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) *n*-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{m\times n}$ (the set of all m×n matrices with real values)

Ex: : ($m = n = 2$)

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) n-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V=c(-\infty,\infty)$ (the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Thm 4.4: (Properties of scalar multiplication)

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{\underset{\uparrow}{(\tfrac{1}{2})}}\underset{\text{integer}}{\underset{\uparrow}{(1)}} = \underset{\text{noninteger}}{\underset{\uparrow}{\tfrac{1}{2}}} \notin V$$ (it is not closed under scalar multiplication)

- Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 8:

  $V=R^2$=the set of all ordered pairs of real numbers

  vector addition: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$

  scalar multiplication: $c(u_1,u_2)=(cu_1,0)$

  Verify V is not a vector space.

  Sol:

  $\because 1(1,1)=(1,0)\neq(1,1)$

  $\therefore$ the set (together with the two given operations) is not a vector space

Keywords in Section 4.2:

- vector space : فضاء متجهات
- *n*-space : فضاء نوني
- matrix space : فضاء مصفوفات
- polynomial space : فضاء متعددات الحدود
- function space : فضاء الدوال

4.3 Subspaces of Vector Spaces

- Subspace:

$(V,+,\bullet)$: a vector space

$$\left.\begin{array}{r} W \neq \phi \\ W \subseteq V \end{array}\right\}$$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Thm 4.5: (Test for a subspace)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- Ex: Subspace of R^2

(1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0)$

(2) Lines through the origin

(3) R^2

- Ex: Subspace of R^3

(1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0,0)$

(2) Lines through the origin

(3) Planes through the origin

(4) R^3

- Ex 2: (A subspace of $M_{2\times 2}$)

  Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Sol:

$\because W \subseteq M_{2\times 2} \qquad M_{2\times 2}$: vector sapces

Let $A_1, A_2 \in W \;\; (A_1^T = A_1, A_2^T = A_2)$

$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \qquad (A_1 + A_2 \in W)$

$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \qquad (kA \in W)$

$\therefore W$ is a subspace of $M_{2\times 2}$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$\therefore W_2$ is not a subspace of $M_{2\times 2}$

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

Show that $W=\{(x_1,x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\mathbf{u}=(1,1)\in W$

$\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1)\notin W$

(not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

Keywords in Section 4.3:

- subspace : فضاء جزئي
- trivial subspace : فضاء جزئي بسيط

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)\\ &= (c_1 - c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)\end{aligned}$$

$$\Rightarrow \begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \ c_2 \qquad\quad &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass−Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$$\Rightarrow \text{ this system has no solution } (\because 0 \neq 7)$$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$span(S)=\{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

- Notes:

$$\text{span}\,(S) = V$$

$$\Rightarrow \; S \text{ spans (generates) } V$$

V is spanned (generated) by S

S is a spanning set of V

- Notes:

(1) $span(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq span(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$$

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$

then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),

then S is called linearly dependent.

- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1, 2, 3)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2},\ \underset{\mathbf{v}_2}{2+5x-x^2},\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e. $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \qquad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S=\left\{\underset{\mathbf{v}_1}{\begin{bmatrix}2&1\\0&1\end{bmatrix}},\underset{\mathbf{v}_2}{\begin{bmatrix}3&0\\2&1\end{bmatrix}},\underset{\mathbf{v}_3}{\begin{bmatrix}1&0\\2&0\end{bmatrix}}\right\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

$$c_1\begin{bmatrix}2&1\\0&1\end{bmatrix}+c_2\begin{bmatrix}3&0\\2&1\end{bmatrix}+c_3\begin{bmatrix}1&0\\2&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

Keywords in Section 4.4:

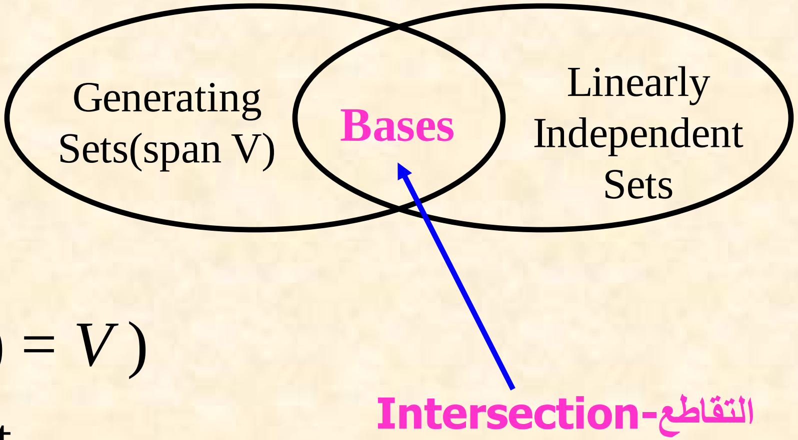
- linear combination : تركيب خطي
- spanning set : مجموعة المدى
- trivial solution : حل بسيط
- linear independent : الاستقلال الخطي
- linear dependent : الاعتماد الخطي

4.5 Basis and Dimension

- Basis:

V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$



If $\begin{cases} (a) & S \text{ spans } V \ \ (\text{i.e., } \ span(S) = V\,) \\ (b) & S \text{ is linearly independent} \end{cases}$

$\Rightarrow$ *Then* S is called a **basis** for V

- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\} \quad i = (1, 0, 0),\ \ j = (0, 1, 0),\ \ k = (0, 0, 1)$

(3) the standard basis for $\boldsymbol{R}^{\boldsymbol{n}}$:

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $m\times n$ matrix space:

$\{ E_{ij} \mid 1\le i\le m , 1\le j\le n \}$

Ex: for $\mathbf{M_{22}}$ 2×2 matrix space:

$$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

(5) the standard basis for $\boldsymbol{P_n(x)}$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

- Thm 4.9: (Uniqueness of basis representation)

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Pf:

$$\because S \text{ is a basis} \Rightarrow \begin{cases} 1. \;\; span(S)=V \\ 2. \;\; S \text{ is linearly independent} \end{cases}$$

$\because span(S)=V$ Let $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$

$\mathbf{v}=b_1\mathbf{v}_1+b_2\mathbf{v}_2+\ldots+b_n\mathbf{v}_n$

$\Rightarrow \mathbf{0}=(c_1-b_1)\mathbf{v}_1+(c_2-b_2)\mathbf{v}_2+\ldots+(c_n-b_n)\mathbf{v}_n$

$\because S$ is linearly independent

$\Rightarrow c_1=b_1,\ c_2=b_2,\ldots,\ c_n=b_n$ (i.e., uniqueness)

- Thm 4.10: (Basis and linear dependence)

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Pf:

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m > n$

$\because span(S)=V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Let $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ (where $d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m$)

$\because S$ is L.I.

$\Rightarrow d_i=0 \quad \forall i$ i.e.

$$\begin{aligned} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m &= 0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m &= 0 \end{aligned}$$

$\because$ According to Thm 1.1: If the homogeneous system has fewer equations than variables, then it must have infinitely many solution.

$m>n \Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

- Thm 4.11: (Number of vectors in a basis)

If a vector space V has one basis with ***n* vectors**, then every basis for V has n vectors. (*All bases for a finite-dimensional vector space has the same number of vectors*.)

Pf:

$S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
$S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ two bases for a vector space

$$\left.\begin{array}{l}\left.\begin{array}{r} S \text{ is a basis} \\ S' \text{ is L.I.}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \geq m \\ \left.\begin{array}{r} S \text{ is L.I.} \\ S' \text{ is a basis}\end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \leq m \end{array}\right\} \Rightarrow n = m$$

- Finite dimensional:

  A vector space V is called **finite dimensional**,

  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,

  then it is called **infinite dimensional**.

- Dimension:

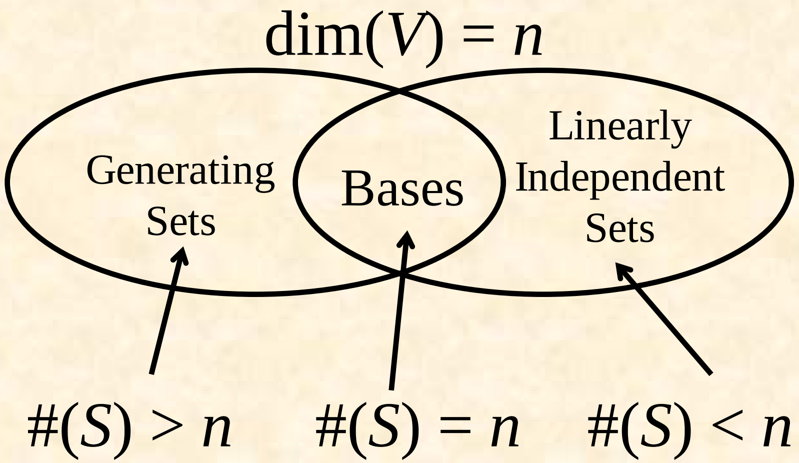
  The **dimension** of a finite dimensional vector space V is defined to be **the number of vectors in a basis** for V.

  V: a vector space S: a basis for V

  $\Rightarrow$symbol: $\mathbf{dim}(V) = \#(S)$ (the number of vectors in S)

- Notes:

(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$



(2) $\dim(V) = n$, $S \subseteq V$

S : a generating set $\Rightarrow$ $\#(S) \geq n$

S : a L.I. set $\Rightarrow$ $\#(S) \leq n$

S : a basis $\Rightarrow$ $\#(S) = n$

(3) $\dim(V) = n$, W is a subspace of V $\Rightarrow$ $\dim(W) \leq n$

- Exp:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m,\ 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

- Ex 9: (Finding the dimension of a subspace)

(a) $W=\{(d, c-d, c)$: c and d are real numbers$\}$

(b) $W=\{(2b, b, 0)$: b is a real number$\}$

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 10: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W=\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a,b,c\in R\right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W $\Rightarrow$ dim(W) = #(S) = 3

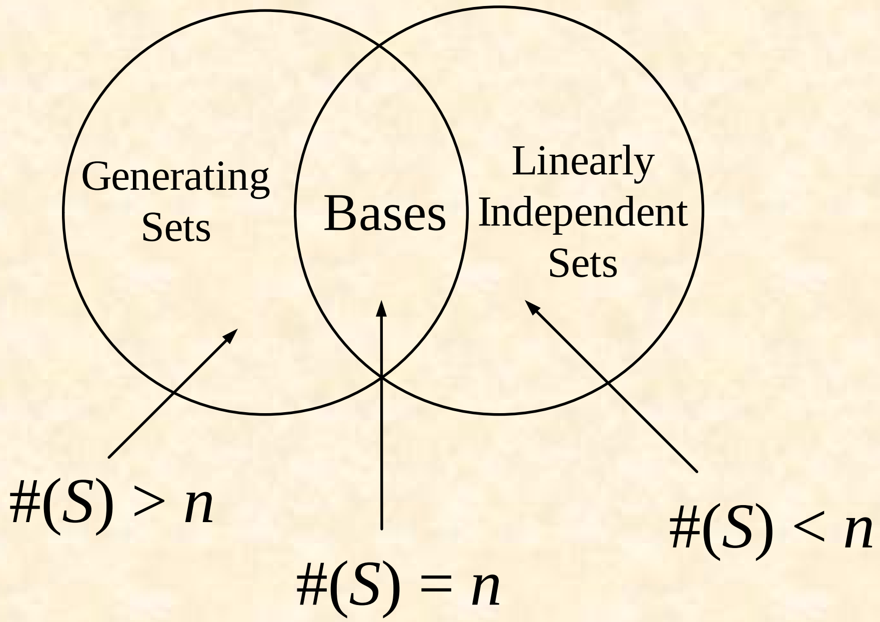
- Thm 4.12: (Basis tests in an n-dimensional space)

Let V be a vector space of **dimension *n***.

(1) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

(2) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ spans V, then S is a basis for V.

$\dim(V) = n$



Imp: If we have a space V of dimension n, and a set of vectors S of number equal n, then for the set of vectors S to be a Basis of V, it is sufficient to show that S is L.I. or that it spans V.

Keywords in Section 4.5:

- Basis : أساس
- Dimension : بعد
- Finite dimension : منتهية البعد
- Infinite dimension : لامنتهية البعد