# Chapter 3
# Determinants

# 3.1 The Determinant of a Matrix

- the determinant of a 2 × 2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) \quad = \quad |A| \quad = \quad a_{11}a_{22} \quad - \quad a_{21}a_{12}$$

- Note:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) \quad = 4 + 3 \quad = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) \quad = 4 - 4 \quad = 0$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) \quad = 0 - 6 \quad = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative.

- Minor of the entry $a_{ij}$ :

The determinant of the matrix determined by deleting the *i*th row and *j*th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of $a_{ij}$ :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- Ex:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21}=(-1)^{2+1}M_{21}=-M_{21} \qquad C_{22}=(-1)^{2+2}M_{22}=M_{22}$$

- Notes: Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

3 × 3 matrix 4 × 4 matrix $n \times n$ matrix

- Notes:

  Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

- Ex 2: Find all the minors and cofactors of *A*.

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix}$$

Sol: (1) All the minors of *A*.

$$\Rightarrow M_{11}=\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-1,\ M_{12}=\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=-5,\ M_{13}=\begin{vmatrix}3 & -1\\ 4 & 0\end{vmatrix}=4$$

$$M_{21}=\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}=2,\quad M_{22}=\begin{vmatrix}0 & 1\\ 4 & 1\end{vmatrix}=-4,\ M_{23}=\begin{vmatrix}0 & 2\\ 4 & 1\end{vmatrix}=-8$$

$$M_{31}=\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix}=5,\quad M_{32}=\begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix}=-3,\ M_{33}=\begin{vmatrix}0 & 2\\ 3 & -1\end{vmatrix}=-6$$

Sol: (2) All the cofactors of $A$.

$$\because \quad C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,\ C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5,\quad C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2,\ C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,\ C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,\ C_{32} = -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3,\quad C_{33} = +\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- Thm 3.1: (Expansion by cofactors)

Let $A$ is a square matrix of order $n$.

Then the determinant of $A$ is given by

$$(a)\quad \det(A)=|A|=\sum_{j=1}^{n} a_{ij}C_{ij}=a_{i1}C_{i1}+a_{i2}C_{i2}+\cdots+a_{in}C_{in}$$

(Cofactor expansion along the $i$-th row, $i$=1, 2,…, n )

or

$$(b)\quad \det(A)=|A|=\sum_{i=1}^{n} a_{ij}C_{ij}=a_{1j}C_{1j}+a_{2j}C_{2j}+\cdots+a_{nj}C_{nj}$$

(Cofactor expansion along the $j$-th row, $j$=1, 2,…, n )

- Ex: The determinant of a matrix of order 3

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13} \\ &= a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23} \\ &= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33} \\ &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31} \\ &= a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32} \\ &= a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33} \end{aligned}$$

- Ex 3: The determinant of a matrix of order 3

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix} \quad \overset{\text{Ex}\,2}{\Rightarrow} \begin{array}{l} C_{11}=-1, C_{12}=5, \quad C_{13}=4 \\ C_{21}=-2, C_{22}=-4, C_{23}=8 \\ C_{31}=5, \quad C_{32}=3, \quad C_{33}=-6 \end{array}$$

Sol:

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}=(0)(-1)+(2)(5)+(1)(4)=14\\ &= a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}=(3)(-2)+(-1)(-4)+(2)(8)=14\\ &= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}=(4)(5)+(0)(3)+(1)(-6)=14\\ &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}=(0)(-1)+(3)(-2)+(4)(5)=14\\ &= a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}=(2)(5)+(-1)(-4)+(0)(3)=14\\ &= a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33}=(1)(4)+(2)(8)+(1)(-6)=14\end{aligned}$$

- Ex 5: (The determinant of a matrix of order 3)

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$C_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-1 \qquad C_{12}=(-1)^{1+2}\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=(-1)(-5)=5$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}3 & -1\\ 4 & 0\end{vmatrix}=4$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}\\ &=(0)(-1)+(2)(5)+(1)(4)\\ &=14\end{aligned}$$

- Notes:

  The row (or column) containing the most zeros is the best choice for expansion by cofactors .

- Ex 4: (The determinant of a matrix of order 4)

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43})$$

$$= 3C_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3\left[ (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

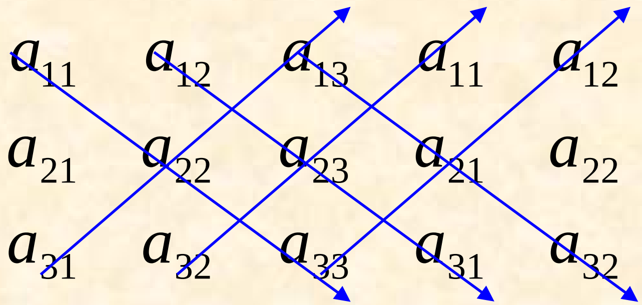
$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Subtract these three products.




Add these three products.

$$\Rightarrow \det(A)=|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}$$
$$-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

- Ex 5:

$$
\begin{array}{ccccc}
 & & -4 & 0 & 6 \\
\end{array}
$$

$$
A=\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}
\begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & -4 \end{matrix}
$$

$$
\begin{array}{ccc}
0 & 16 & -12 \\
\end{array}
$$

$$
\Rightarrow \det(A) = |A| = 0+16-12-(-4)-0-6=2
$$

- Upper triangular matrix:

All the entries below the main diagonal are zeros.

- Lower triangular matrix:

All the entries above the main diagonal are zeros.

- Diagonal matrix:

All the entries above and below the main diagonal are zeros.

- Note:

A matrix that is both upper and lower triangular is called diagonal.

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular    lower triangular    diagonal

- Thm 3.2: (Determinant of a Triangular Matrix)

If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

- Ex 6: Find the determinants of the following triangular matrices.

$$(a)\quad A=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \qquad (b)\quad B=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$(a)\quad |A| = (2)(-2)(1)(3) = -12$

$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$

# Keywords in Section 3.1:

- determinant : المحدد
- minor : المختصر
- cofactor : المعامل
- expansion by cofactors : التحليل بالمعاملات
- upper triangular matrix: مصفوفة مثلثية سفلى
- lower triangular matrix: مصفوفة مثلثية عليا
- diagonal matrix: مصفوفة قطرية

# 3.2 Evaluation of a determinant using elementary operations

- Thm 3.3: (Elementary row operations and determinants)

  Let $A$ and $B$ be square matrices.

  $(a)\quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e.} \left| r_{ij}(A) \right| = -\left| A \right|)$

  $(b)\quad B = r_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e.} \left| r_{i}^{(k)}(A) \right| = k\left| A \right|)$

  $(c)\quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e.} \left| r_{ij}^{(k)}(A) \right| = \left| A \right|)$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \qquad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \quad \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \quad \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \quad \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always upper triangular.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A=\begin{bmatrix}2 & -3 & 10\\ 1 & 2 & -2\\ 0 & 1 & -3\end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$\det(A)=\begin{vmatrix}2 & -3 & 10\\ 1 & 2 & -2\\ 0 & 1 & -3\end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix}1 & 2 & -2\\ 2 & -3 & 10\\ 0 & 1 & -3\end{vmatrix}$$

$$\overset{r_{12}^{(-2)}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_{23}^{(-1)}}{=} 7 \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = (7)(1)(1)(-1) = -7$$

- Notes:

$$|EA| = |E||A|$$

(1) $E = R_{ij}$ $\Rightarrow |E| = |R_{ij}| = -1$

$$\Rightarrow |EA| = |r_{ij}(A)| = -|A| = |R_{ij}||A| = |E||A|$$

(2) $E = R_i^{(k)}$ $\Rightarrow |E| = |R_i^{(k)}| = k$

$$\Rightarrow |EA| = |r_i^{(k)}(A)| = k|A| = |R_i^{(k)}||A| = |E||A|$$

(3) $E = R_{ij}^{(k)}$ $\Rightarrow |E| = |R_{ij}^{(k)}| = 1$

$$\Rightarrow |EA| = |r_{ij}^{(k)}(A)| = 1|A| = |R_{ij}^{(k)}||A| = |E||A|$$

- Determinants and elementary column operations

- Thm: (Elementary column operations and determinants)

  Let $A$ and $B$ be square matrices.

$$(a) \quad B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b) \quad B = c_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |c_{i}^{(k)}(A)| = k|A|)$$

$$(c) \quad B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

- Ex:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \quad \Rightarrow \det(A_1) = \det(c_1^{(4)}(A)) = \frac{1}{2}\det(A) = (\frac{1}{2})(-8) = -4$$

$$A_2 = c_{12}(A) \quad \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \quad \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- Thm 3.4: (Conditions that yield a zero determinant)

If $A$ is a square matrix and any of the following conditions is true, then det $(A) = 0$.

($a$) An entire row (or an entire column) consists of zeros.

($b$) Two rows (or two columns) are equal.

($c$) One row (or column) is a multiple of another row (or column).

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Note:

| Order $n$ | Cofactor Expansion | | Row Reduction | |
|---|---|---|---|---|
| | Additions | Multiplications | Additions | Multiplications |
| 3 | 5 | 9 | 5 | 10 |
| 5 | 119 | 205 | 30 | 45 |
| 10 | 3,628,799 | 6,235,300 | 285 | 339 |

- Ex 5: (Evaluating a determinant)

$$A=\begin{bmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{C_{13}^{(2)}}{=}\begin{vmatrix}-3 & 5 & -4\\ 2 & -4 & 3\\ -3 & 0 & 0\end{vmatrix}$$

$$=(-3)(-1)^{3+1}\begin{vmatrix}5 & -4\\ -4 & 3\end{vmatrix}=(-3)(-1)=3$$

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{r_{12}^{(\frac{4}{5})}}{=}\begin{vmatrix}-3 & 5 & 2\\ \frac{-2}{5} & 0 & \frac{3}{5}\\ -3 & 0 & 6\end{vmatrix}$$

$$=(5)(-1)^{1+2}\begin{vmatrix}\frac{-2}{5} & \frac{3}{5}\\ -3 & 6\end{vmatrix}=(-5)(-\frac{3}{5})=3$$

- Ex 6: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{r_{25}^{(-1)}}{\overset{}{\underset{r_{24}^{(1)}}{=}}} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{c_{41}^{(-3)}}{=} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \underset{r_{21}^{(1)}}{=} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

# 3.3 Properties of Determinants

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22}+b_{22} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: (The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

- Check:

$$|AB| = |A|\,|B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If $A$ is an $n \times n$ matrix and $c$ is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Thm 3.7: (Determinant of an invertible matrix)

A square matrix $A$ is invertible (nonsingular) if and only if

$$\det(A) \neq 0$$

- Ex 3: (Classifying square matrices as singular or nonsingular)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \quad \Rightarrow$ B has an inverse (it is nonsingular).

- Thm 3.8: (Determinant of an inverse matrix)

$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

- Thm 3.9: (Determinant of a transpose)

$$\text{If } A \text{ is a square matrix, then } \det(A^{T}) = \det(A).$$

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{(a) } \left|A^{-1}\right| = ? \qquad \text{(b) } \left|A^{T}\right| = ?$$

Sol:

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \qquad \therefore \quad \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

(1) $A$ is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) $A$ is row-equivalent to $I_n$

(5) $A$ can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

$$(a)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

(a) $A\mathbf{x}=\mathbf{b}$

$\because \quad |A|=0$

$\therefore$ This system does not have a unique solution.

(b) $B\mathbf{x}=\mathbf{b}$

$\because \quad |B|=-12\neq 0$

$\therefore$ This system has a unique solution.

# 3.4 Introduction to Eigenvalues

- Eigenvalue problem:

  If $A$ is an $n \times n$ matrix, do there exist $n \times 1$ nonzero matrices $\boldsymbol{x}$ such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$ ?

- Eigenvalue and eigenvector:

  $A$ : an $n \times n$ matrix

  $\lambda$ : a scalar

  $\boldsymbol{x}$ : a $n \times 1$ nonzero column matrix

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

(The fundamental equation for the eigenvalue problem)

- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix} \qquad x_1=\begin{bmatrix}1\\ 1\end{bmatrix} \qquad x_2=\begin{bmatrix}2\\ -1\end{bmatrix}$$

Eigenvalue

$$Ax_1=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}5\\ 5\end{bmatrix}=5\begin{bmatrix}1\\ 1\end{bmatrix}=5x_1$$

Eigenvector

Eigenvalue

$$Ax_2=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}-2\\ 1\end{bmatrix}=-1\begin{bmatrix}2\\ -1\end{bmatrix}=(-1)x_2$$

Eigenvector

- Question:

  Given an $n \times n$ matrix $A$, how can you find the eigenvalues and corresponding eigenvectors?

- Note:

  $$Ax = \lambda x \;\Rightarrow\; (\lambda \mathrm{I} - A)x = 0$$ (homogeneous system)

  If $(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$

- Characteristic equation of $A \in M_{n \times n}$:

  $$\det(\lambda \mathrm{I} - A) = \left|(\lambda \mathrm{I} - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

- Ex 2: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 5, -1$$

Eigenvalues: $\lambda_1 = 5, \lambda_2 = -1$

$$(1)\lambda_1 = 5 \qquad \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; t \neq 0$$

$$(2)\lambda_2 = -1 \qquad \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \; t \neq 0$$

- Ex 3: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -1 & \lambda - 2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1)(\lambda - 3) = 0$$

$$\text{The eigenvalues}: \ \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 3$$

$$\lambda_1 = 1$$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors:} \quad t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = -1$$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -2 & -2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors:} \quad t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 3$$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & -2 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors:} \quad t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

# 3.4 Applications of Determinants

- Matrix of cofactors of $A$:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of $A$:

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Thm 3.10: (The inverse of a matrix given by its adjoint)

  If $A$ is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Ex 1 & Ex 2:

$$A=\begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

(*a*) Find the adjoint of *A*.

(*b*) Use the adjoint of *A* to find $A^{-1}$

Sol: $\because \quad C_{ij} = (-1)^{i+j} M_{ij}$

$$\Rightarrow C_{11} = +\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4,\ C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1,\ C_{13} = +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = 6,\quad C_{22} = +\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 0,\ C_{23} = -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$C_{31} = +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 7,\quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1,\ C_{33} = +\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

$\Rightarrow$ cofactor matrix of $A$ $\Rightarrow$ adjoint matrix of $A$

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \qquad adj(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of $A$

$$A^{-1} = \frac{1}{\det(A)} adj(A) \qquad \because \quad \det(A) = 3$$

$$= \tfrac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

- Check: $AA^{-1} = I$

- Thm 3.11: (Cramer's Rule)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b} \qquad A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}, A^{(2)}, \cdots, A^{(n)}\right] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{ i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1, 2, \cdots, n$$

- Pf:

$$A\mathbf{x} = \mathbf{b}, \qquad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \quad = \frac{1}{\det(A)} adj(A)\mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & \vdots & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow x_j = \frac{1}{\det(A)}(b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj})$$

$$= \frac{\det(A_j)}{\det(A)} \qquad j = 1,2,\cdots,n$$

- Ex 4: Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \quad\quad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

# Keywords in Section 3.4:

- matrix of cofactors : مصفوفة المعاملات
- adjoint matrix : مصفوفة مصاحبة
- Cramer's rule : قانون كرامر