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Research paper

Multiple attribute decision-making model for artificially intelligent last-mile delivery robots selection in neutrosophic square root environment

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ABSTRACT

We introduce novel methodological techniques for decision-making with multiple attributes utilizing logarithmic square root neutrosophic vague sets. One important thing is that we improved decision-making by adding logarithmic square root neutrosophic ambiguous weighted operators. Logarithmic square root, neutrosophic imprecise weighted averaging, geometric procedures, and expanded versions of these are some of the data processing methodologies that we explore. The use of Hamming distances and Euclidean distances in decision-making situations is illustrated by real-world instances. To clarify the basic properties of these sets, the research uses an algebraic framework. Numerous domains make use of neural networks, including translation, medical diagnosis, and picture and speech recognition. Developing multipurpose artificially intelligent robots with analytical, functional, visual, interactive, and textual capabilities relies heavily on the synergy between computer science and machine tool technology. This is especially true when it comes to the evolution of artificial intelligence. The operating procedures, expenses, time, and externalizes of an artificially intelligent robot system should be considered while assessing its quality. Finding the best answer from a list of possibilities is made easier with the help of expert views and established criteria. By comparing them to other methods, we verify and show that the suggested models work. The study's findings highlight the importance of the research.

1. Introduction

Robotics has its roots in the early industrial era, when basic automation devices were created to carry out repetitive and ordinary activities. More developments in control engineering and computers in the ensuing decades made it possible to create increasingly complex robots that could carry out a larger variety of activities. George Devol and Joseph Engelberger unveiled the first industrial robot in 1956, which was a huge turning point in the history of robotics. Robots became a common tool in the manufacturing sector by the 1970s, especially in the manufacturing of automobiles. The subsequent decades saw the creation of more compact and adaptable robots that could be employed in a variety of industries thanks to developments in electronic and software engineering as well as shrinking. Modern robotics is developing quickly thanks to the combination of machine learning (ML) and artificial intelligence (AI), which allows robots to interact with their surroundings on their own and carry out increasingly complicated tasks. The most well-known sectors using robotics extensively include manufacturing, healthcare, logistics, and agriculture (Vijayakuymar and Suresh, 2022; Oliveira et al., 2021). Robots can now evaluate enormous volumes of data, draw lessons from their experiences, and modify their actions as necessary thanks to AI and ML algorithms (Soori et al., 2023). A comprehensive evaluation of current work on ML and human–robot collaboration (HRC) was carried out by Semeraro et al. (2023) A thorough assessment of the literature on the applications and outcomes of

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Abbreviations

DM	Decision making
MADM	Multiple attribute decision-making
FS	Fuzzy set
IVFS	Interval-valued fuzzy set
IFS	Intuitionistic fuzzy set
PyFS	Pythagorean fuzzy set
PFS	Picture fuzzy set
IVPFS	Interval-valued Pythagorean fuzzy set
SFS	Spherical fuzzy set
NS	Neutrosophic set
MG	Membership grade
NMG	Non-membership grade
TMG	Truth membership grade
FMG	False membership grade
ED	Euclidean distance
HD	Hamming distance
VS	Vague set
LSRNSVS	Logarithmic square root neutrosophic vague set
LSRNSVN	Logarithmic square root neutrosophic vague number
LSRNSVWA	Logarithmic square root neutrosophic vague weighted averaging
LSRNSVWG	Logarithmic square root neutrosophic vague weighted geometric
LSRGNSVWA	Logarithmic square root generalized neutrosophic vague weighted averaging
LSRGNSVWG	Logarithmic square root generalized neutrosophic vague weighted geometric

intelligent physical robots in the healthcare industry was finished by Huang et al. (2023). Robots are employed in a variety of production processes because of their accuracy, consistency, and speed, including welding, assembling, and packaging. Another industry where robots can be extremely helpful to medical personnel in diagnosing, operating, and recovering from injuries is healthcare. Robots can be used in agriculture for a variety of tasks, such as planting, harvesting, and fertilizing crops. They offer benefits like cost-effectiveness, efficiency and precision.

Kaplan and Haenlein (2020) discussed the concept of AI. The concept of major economies engaging in substantial policy activities to promote AI research and development is explained by Margetts and C (2019). Among the key technical subsystems that construct the current AI technological paradigm are machine learning, neural networks, natural language processing (NLP), smart robots, knowledge graphs, and expert systems (Cresswell et al., 2020; Yablonsky, 2019). In addition to ethical concerns, AI also poses a risk to society in terms of the impacts it will have on democracy and the labor market. Technology assessment (TA) activities based on interdisciplinary TA are necessary to assess these risks and opportunities. A number of studies have also claimed that AI can have an impact on all economic and social sectors, making it a general-purpose technology. The development of deep learning technology will not only allow online platforms to reap market profits, but also provide social governance tools that are highly efficient. Moreover, AI technology has the potential to transform social structures as it becomes increasingly applied in specific economic and financial domains (Klinger et al., 2018; Rasskazov, 2020).

As real-world systems continually evolve, it can be challenging for decision-makers to choose the best course of action. It is possible to reduce a number of goals to a single one, despite the difficulty. The restriction of people's motivations, goals, and viewpoints was difficult for many businesses. While making decisions, people or committees must consider multiple goals at once. Based on this view, decision-makers are prevented from selecting the best course of action, the one that meets all practical requirements. Therefore, more practical and reliable methods are developed for identifying the best option for decision-makers. Agostini et al. (2017) discussed the concept of an efficient interactive decision-making (DM) framework for robotic applications. Many authors deal with the level of harmful emissions from courier companies and their impact on the environment (Lazarević et al., 2020; Lazarević and Dobrodolac, 2020). Fulfilling the high expectations of customers is very challenging, therefore some authors propose meta-heuristic algorithms to be used as a support in optimizing last-mile assignment problems (Zhang et al., 2022). Choosing the appropriate vehicles and transportation modes for last-mile delivery is also an intricate question recently analyzed in the literature (Simić et al., 2021; Švadlenka et al., 2020). Through technology development, we are in a position today to introduce vehicles based on the implementation of artificial intelligence, i.e. last-mile delivery robots. In the literature, such robots are also called autonomous delivery vehicles (Lu et al., 2023; Lazarević et al., 2023).

Decision-makers find it increasingly difficult to identify the optimal solution as real-world systems become increasingly complex. Selecting the best option is possible despite the difficulty of deciding between alternatives. Opportunities, objectives, and viewpoint constraints are challenging to create for many firms. In line with this, when DM, individuals or groups should consider multiple objectives at the same time. A wide variety of MADM-related issues are dealt with every day. Our DM abilities need to be improved as a result. This field of study has been studied by a variety of researchers using a range of methods. There are several uncertain theories proposed by them to deal with the uncertainties, including fuzzy set (FS) (Zadeh, 1965), intuitionistic fuzzy set (IFS) (Atanassov, 1986), interval valued FS (IVFS) (Gorzalczany, 1987), vague set (Biswas, 2006), Pythagorean fuzzy set (PyFS) (Yager, 2014), IVPFS (Peng and Yang, 2015), spherical FS (SFS) (Ashraf et al., 2019). A membership grade (MG) indicates how well an FS fits into the specified set ranging from 0 to 1. An IFS concept was later introduced by Atanassov (1986), in which each object has two MGs positive ξ and negative β and satisfies $0 \leq \xi + \beta \leq 1$, for $\xi, \beta \in [0, 1]$. Yager (2014) developed the concept of PyFSs, which are defined by their MGs and grade of non-memberships (NMGs) under the condition that $\xi + \beta \geq 1$ to $\xi^2 + \beta^2 \leq 1$. Extensive research has been conducted on the implementation of IFSs and PyFSs in several fields. They still have limited skills in expressing information. Consequently, the experts still had difficulties conveying the information in these sets and their associated information. The concept of a picture fuzzy set (PFS) was developed by Cuong and Kreinovich to overcome this information. Therefore, it has been noted that PFS is an expanded version of IFS that can accommodate additional ambiguities. In PFS, it was observed that MG ξ, neutral ζ and non-MG ν with $0 \leq \xi+\zeta+\nu \leq 1$; for $\xi, \zeta, \nu \in [0,1]$. Using the PFS definition will ensure that expert opinions are conveyed, such as "yes", "abstain", "no", and "refusal", while also avoiding missing evaluation details and encouraging the consistency of the acquired information between the actual decision environment and the evaluation data. There are many applications and studies of PFS, but its concept has not been widely studied. Ashraf et al. (2019) defined the spherical fuzzy set (SFS) for some AOs with MADM. In SFS, under the condition that $0 \leq \xi^2+\zeta^2+\nu^2 \leq 1$ rather than $0 \leq \xi+\zeta+\nu \leq 1$. Linguistic spherical fuzzy AOs were proposed by Jin et al. (2019) and discussed in MADM problems. SFSs and their applications in DM were introduced by Rafiq et al. (2019). A DM problem is a property in which $\xi^2+\zeta^2 \geq 1$. Senapati and Yager (2020) proposed the concept of FFS in 2019. A characteristic feature of MG and NMG is that their $0 \leq \xi^3+\zeta^3 \leq 1$. A geometric AO on interval-valued PyFS (IVPFS) was discussed by Rahman et al. (2017). Rahman et al. (2018) introduced the concept of IVPFS using Einstein AO as an effective method for MAGDM. Peng and Yang (2015) interacted interval valued PyFSs with aggregation operators (AOs).

Yager (2014) developed PyFS, which is characterized by a square sum of its MG and NMG not exceeding one. In order to generalize IFS, Yager used PyFS to build a model. A new concept was proposed by Yager (2016) in light of society's continuous complexity and theory development. The MG and NMG in the q-rung orthogonal pair FS (q-ROFS) have power q, but the sum can never exceed one. The IFSs and PyFSs can all be considered special cases of q-ROFSs, therefore they are general. The use of q-ROFSs can thus express fuzzy information in a broader range. Because the parameter q can be adjusted, q-ROFSs are flexible and better suited to uncertain environments. An increase in q can be made as ambiguity in decision information increases. It is possible that some experts are influenced by both their own desires and their surroundings. Therefore, they may have an MDG of 0.95 and an NMG of 0.55 when evaluating certain DM things. The fuzzy information cannot be described by IFNs and PFNs, but q-ROFNs can be described if parameter q is increased. Due to this, the q-ROFS is more flexible and suitable for describing uncertain data. IVPyFSs were created by Yang et al. in conjunction with the MADM (Yang and Chang, 2020) aggregating operations. Al-shami et al. (2022) examined square root FS (SRFS) and its weighted aggregated operators in the context of DM.

Smarandache (1999) introduced a new theory called the NSS. IFSs and FSs differ from each other in that they are both neutral cognitive systems. Neutosophy studies neutral cognition. Accordingly, the truth degree (TD), the indeterminacy degree (ID), and the falsehood degree (FD) are assigned to each assertion. For each of the components of the universe in the NSS set, the degree of TD, ID, and FD falls into the range of $[0,1]$. It has been shown from a philosophical standpoint that an NSS can generalize a classical set, an FS, an IFS, etc. Smarandache (1999) invented the Pythagorean NSIV set (PNSIVS). Vagues sets (VSs) were introduced in Biswas (2006). There are two functions in VSs, a truth-membership function t_v and a false-membership function f_v. Here, $t_v(x)$ represents the lower bound on the grade of membership for x, and $f_v(x)$ represents the lower bound on the negation for x derived from the evidence against x. In $[0,1]$, $t_v(x)$ and $f_v(x)$ which sum does not exceed 1. VSs have a number of proven applications (Bustince and Burillo, 1996; Kumar et al., 2007; Wang et al., 2006). Ejegwa (2018) discussed and extended distance metrics for IFSs, such as Hamming distances (HDs), Euclidean distances (EDs), normalized HDs, and normalized EDs. Palanikumar et al. (2022) investigated the Pythagorean NSNV AO using MADM. We see that the majority of the distance functions for PNSNIVSs are generalized in the Pythagorean NSIV set (PNSIVS).

According to Zhang and Xu (2014), PyFS based on TOPSIS should be generalized to include MCDM. A number of practical MADM problems were discussed by Hwang and Yoon (1981). Using MCDM with bipolar FS, Jana and Pal (2019) proposed a robust technique for handling n-valued single-valued soft sets. Ullah et al. (2019) for their description of distance measuring for sophisticated PyFS with practical applications in pattern recognition. A recently presented MCDM approach using single-valued trigonometric number (SVTrN) mappings was developed by Jana and Pal (2021) using a new strategy for NSS dombi power AOs. Palanikumar et al. (2023) developed the medical robotic engineering management in the interval neutrosophic square root approach. Palanikumar et al. (2024) discussed the concept of q-rung vague sets using multi-criteria decision-making. Abed et al. (2023) studied neutrosophic relations in group theory.

The goal of this study is to clarify the significance of LSRNSVS data. We use aggregation procedures to extract data from LSRNSVS. As an example, we will create a rating with these operators and apply it to DM issues. Consequently, the work's main contributions are as follows:

1. Several algebraic properties of LSRNSVS have been established, such as associativity, distributivity, and idempotency.
2. Am LSRNSVN is characterized by HD and ED. The purpose of this method is to calculate the ED distance between two LSRNSVSs. In addition, we have discussed the idea of converting LSRNSVNs into NFNs as well.
3. It is the purpose of this study to illustrate numerically how LSRNSVNs can be used to apply MADMs and AOs in real-world scenarios. The LSRNSVN requires an algorithm to be developed. In addition, a normalized decision matrix must be determined by applying LSRNSVNs to the response matrix.
4. A value must be derived for each concept discussed in the LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG.
5. The decision-maker can now use LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG operators in a flexible manner to choose the ranking result based on their preference. As a result of its many advantages, we propose an extremely flexible method in this paper.

The seven sections of the paper are listed below. The introduction is located in Section 1. Section 2 provides a brief overview of the ideas involved. The LSRNSVN-based MADM and its processes are discussed in Section 3. The distance between LSRNSVNs is used by Section 4 to communicate over MADM. The MADM for LSRNSVN based on a few aggregating processes is discussed in Section 5. There is a mathematical example, analysis, and discussion of the MADM algorithm based on LSRNSV data in Section 6. There is a conclusion in Section 7.

2. Preliminaries

The following section will briefly review some key concepts related to our future studies.

Definition 1 (*Yager, 2014*)**.** Let $\mathbb{X}$ be the universe, PyFS $\chi = \left\{\varrho, \left\langle \Delta^{\mathbb{T}}_{\chi}(\varrho), \Delta^{\mathbb{F}}_{\chi}(\varrho)\right\rangle \middle| \varrho \in \mathbb{X}\right\}$, $\Delta^{\mathbb{T}}_{\chi} : \mathbb{X} \to [0,1]$ and $\Delta^{\mathbb{F}}_{\chi} : \mathbb{X} \to [0,1]$ are denotes the MD and NMD of $\varrho \in \mathbb{X}$ to χ, respectively and $0 \leq (\Delta^{\mathbb{T}}_{\chi}(\varrho))^2 + (\Delta^{\mathbb{F}}_{\chi}(\varrho))^2 \leq 1$. For $\chi = \left\langle \Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right\rangle$ is called a Pythagorean fuzzy number (PyFN).

Definition 2 (*Al-shami et al., 2022*)**.** The SRFS $\chi = \left\{\varrho, \left\langle \Delta^{\mathbb{T}}_{\chi}(\varrho), \Delta^{\mathbb{F}}_{\chi}(\varrho)\right\rangle \middle| \varrho \in \mathbb{X}\right\}$, $\Delta^{\mathbb{T}}_{\chi} : \mathbb{X} \to [0,1]$ and $\Delta^{\mathbb{F}}_{\chi} : \mathbb{X} \to [0,1]$ are denotes the MD and NMD of $\varrho \in \mathbb{X}$ to χ, respectively and $0 \leq (\Delta^{\mathbb{T}}_{\chi}(\varrho))^2 + \sqrt{\Delta^{\mathbb{F}}_{\chi}(\varrho)} \leq 1$. For $\chi = \left\langle \Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right\rangle$ is called a square root fuzzy number (SRFN).

Definition 3 (Peng and Yang, 2015). The PyIVFS $\overline{\chi} = \left\{ \varrho, \left\langle \overline{\Delta^{\mathbb{T}}_{\chi}}(\varrho), \overline{\Delta^{\mathbb{F}}_{\chi}}(\varrho) \right\rangle \middle| \varrho \in \mathbb{X} \right\}$, where $\overline{\Delta^{T}_{\chi}} : \mathbb{X} \to Int([0,1])$ and $\overline{\Delta^{F}_{\chi}} : \mathbb{X} \to Int([0,1])$ are denotes the MD and NMD of $\varrho \in \mathbb{X}$ to χ, respectively, and $0 \le (\Delta^{\mathbb{T}u}_{\chi}(\varrho))^2 + (\Delta^{\mathbb{F}u}_{\chi}(\varrho))^2 \le 1$. For $\overline{\chi} = \left\langle \left[\Delta^{\mathbb{T}l}_{\chi}, \Delta^{\mathbb{T}u}_{\chi}\right], \left[\Delta^{\mathbb{F}l}_{\chi}, \Delta^{\mathbb{F}u}_{\chi}\right] \right\rangle$ is called a Pythagorean interval-valued fuzzy number (PyIVFN).

Definition 4 (Peng and Yang, 2015). Let $\overline{\chi} = \left\langle [\Delta^{\mathbb{T}l}, \Delta^{\mathbb{T}u}], [\Delta^{\mathbb{F}l}, \Delta^{\mathbb{F}u}] \right\rangle$, $\overline{\chi}_1 = \left\langle [\Delta^{\mathbb{T}l}_1, \Delta^{\mathbb{T}u}_1], [\Delta^{\mathbb{F}l}_1, \Delta^{\mathbb{F}u}_1] \right\rangle$ and $\overline{\chi}_2 = \left\langle [\Delta^{\mathbb{T}l}_2, \Delta^{\mathbb{T}u}_2], [\Delta^{\mathbb{F}l}_2, \Delta^{\mathbb{F}u}_2] \right\rangle$ be the PyIVFNs, and $\Gamma > 0$. Then,

1. $\overline{\chi}_1 \bigwedge \overline{\chi}_2 = \left[\begin{array}{c} \left[\sqrt{(\Delta^{\mathbb{T}l}_1)^2 + (\Delta^{\mathbb{T}l}_2)^2 - (\Delta^{\mathbb{T}l}_1)^2 \cdot (\Delta^{\mathbb{T}l}_2)^2}, \sqrt{(\Delta^{\mathbb{T}u}_1)^2 + (\Delta^{\mathbb{T}u}_2)^2 - (\Delta^{\mathbb{T}u}_1)^2 \cdot (\Delta^{\mathbb{T}u}_2)^2} \right], \\ \left[\Delta^{\mathbb{F}l}_1 \cdot \Delta^{\mathbb{F}l}_2, \Delta^{\mathbb{F}u}_1 \cdot \Delta^{\mathbb{F}u}_2 \right] \end{array} \right]$,
2. $\overline{\chi}_1 \bigvee \overline{\chi}_2 = \left[\begin{array}{c} \left[\Delta^{\mathbb{T}l}_1 \cdot \Delta^{\mathbb{T}l}_2, \Delta^{\mathbb{T}u}_1 \cdot \Delta^{\mathbb{T}u}_2 \right], \\ \left[\sqrt{(\Delta^{\mathbb{F}l}_1)^2 + (\Delta^{\mathbb{F}l}_2)^2 - (\Delta^{\mathbb{F}l}_1)^2 \cdot (\Delta^{\mathbb{F}l}_2)^2}, \sqrt{(\Delta^{\mathbb{F}u}_1)^2 + (\Delta^{\mathbb{F}u}_2)^2 - (\Delta^{\mathbb{F}u}_1)^2 \cdot (\Delta^{\mathbb{F}u}_2)^2} \right] \end{array} \right]$,
3. $\Gamma \cdot \overline{\chi} = \left[\left[\sqrt{1 - (1 - (\Delta^{\mathbb{T}l})^2)^{\Gamma}}, \sqrt{1 - (1 - (\Delta^{\mathbb{T}u})^2)^{\Gamma}} \right], \left[(\Delta^{\mathbb{F}l})^{\Gamma}, (\Delta^{\mathbb{F}u})^{\Gamma} \right] \right]$,
4. $\overline{\chi}^{\Gamma} = \left[\left[(\Delta^{\mathbb{T}l})^{\Gamma}, (\Delta^{\mathbb{T}u})^{\Gamma} \right], \left[\sqrt{1 - (1 - (\Delta^{\mathbb{F}l})^2)^{\Gamma}}, \sqrt{1 - (1 - (\Delta^{\mathbb{F}u})^2)^{\Gamma}} \right] \right]$.

Definition 5 (Peng and Yang, 2015). For any PyIVFN $\overline{\chi} = \left\langle [\Delta^{\mathbb{T}l}, \Delta^{\mathbb{T}u}], [\Delta^{\mathbb{F}l}, \Delta^{\mathbb{F}u}] \right\rangle$, the score function of $\overline{\chi}$ is

$$S(\overline{\chi}) = \frac{1}{2}\left((\Delta^{\mathbb{T}l})^2 + (\Delta^{\mathbb{T}u})^2 - (\Delta^{\mathbb{F}l})^2 - (\Delta^{\mathbb{F}u})^2 \right), \; S(\overline{\chi}) \in [-1,1],$$

Accuracy function of $\overline{\chi}$ is

$$H(\overline{\chi}) = \frac{1}{2}\left((\Delta^{\mathbb{T}l})^2 + (\Delta^{\mathbb{T}u})^2 + (\Delta^{\mathbb{F}l})^2 + (\Delta^{\mathbb{F}u})^2 \right), \; H(\overline{\chi}) \in [0,1].$$

Definition 6. For any SRIVFN $\overline{\chi} = \left\langle [\Delta^{\mathbb{T}l}, \Delta^{\mathbb{T}u}], [\Delta^{\mathbb{F}l}, \Delta^{\mathbb{F}u}] \right\rangle$, the score function of $\overline{\chi}$ is

$$S(\overline{\chi}) = \frac{1}{2}\left((\Delta^{\mathbb{T}l})^2 + (\Delta^{\mathbb{T}u})^2 - \sqrt{\Delta^{\mathbb{F}l}} - \sqrt{\Delta^{\mathbb{F}u}} \right), \; S(\overline{\chi}) \in [-1,1],$$

Accuracy function of $\overline{\chi}$ is

$$H(\overline{\chi}) = \frac{1}{2}\left((\Delta^{\mathbb{T}l})^2 + (\Delta^{\mathbb{T}u})^2 + \sqrt{\Delta^{\mathbb{F}l}} + \sqrt{\Delta^{\mathbb{F}u}} \right), \; H(\overline{\chi}) \in [0,1].$$

Definition 7 (Smarandache, 1999). The NSS $\chi = \left\{ \varrho, \left\langle \varrho^{\mathbb{T}}_{\chi}(\varrho), \varrho^{\mathbb{I}}_{\chi}(\varrho), \varrho^{\mathbb{F}}_{\chi}(\varrho) \right\rangle \middle| \varrho \in \mathbb{X} \right\}$, where $\varrho^{\mathbb{T}}_{\chi} : \mathbb{X} \to [0,1]$, $\varrho^{\mathbb{I}}_{\chi} : \mathbb{X} \to [0,1]$ and $\varrho^{\mathbb{F}}_{\chi} : \mathbb{X} \to [0,1]$ denote the TD, ID and FD of $\varrho \in \mathbb{X}$ to χ, respectively and $0 \le \varrho^{\mathbb{T}}_{\chi}(\varrho) + \varrho^{\mathbb{I}}_{\chi}(\varrho) + \varrho^{\mathbb{F}}_{\chi}(\varrho) \le 2$. For $\chi = \langle \varrho^{\mathbb{T}}_{\chi}, \varrho^{\mathbb{I}}_{\chi}, \varrho^{\mathbb{F}}_{\chi} \rangle$ is called a neutrosophic fuzzy number (NSFN).

Definition 8 (Biswas, 2006). (i) A VS χ_1 in $\mathbb{X}$ is a pair $(\mathbb{T}_{\chi_1}, \mathbb{F}_{\chi_1})$, $\mathbb{T}_{\chi_1}, \mathbb{F}_{\chi_1} : \mathbb{X} \to [0,1]$ are mappings and $\mathbb{T}_{\chi_1}(\varrho) + \mathbb{F}_{\chi_1}(\varrho) \le 1$, for all $\varrho \in \mathbb{X}$, $\mathbb{T}_{\chi_1}$ and $\mathbb{F}_{\chi_1}$ are called TD and FD, respectively. (ii) $\chi_1(\varrho) = [\mathbb{T}_{\chi_1}(\varrho), 1 - \mathbb{F}_{\chi_1}(\varrho)]$ is called the vague value of ϱ in χ_1.

Definition 9 (Biswas, 2006). (i) The two VSs are χ_1 and χ_2. Then $\chi_1 \subseteq \chi_2$ if and only if $\chi_1(\varrho) \le \chi_{1\chi_2}(\varrho)$. That is, $\mathbb{T}_{\chi_1}(\varrho) \le \mathbb{T}_{\chi_2}(\varrho)$ and $1 - \mathbb{F}_{\chi_1}(\varrho) \le 1 - \mathbb{F}_{\chi_2}(\varrho)$, for all $\varrho \in \mathbb{X}$.
(ii) The union of two VSs χ_1 and χ_2, as $X = \chi_1 \cup \chi_2$, $\mathbb{T}_X = max\{\mathbb{T}_{\chi_1}, \mathbb{T}_{\chi_2}\}$ and $1 - \mathbb{F}_X = max\{1 - \mathbb{F}_{\chi_1}, 1 - \mathbb{F}_{\chi_2}\} = 1 - min\{\mathbb{F}_{\chi_1}, \mathbb{F}_{\chi_2}\}$.
(iii) The intersection of two VSs χ_1 and χ_2 as $X = \chi_1 \cap \chi_2$, $\mathbb{T}_X = min\{\mathbb{T}_{\chi_1}, \mathbb{T}_{\chi_2}\}$ and $1 - \mathbb{F}_X = min\{1 - \mathbb{F}_{\chi_1}, 1 - \mathbb{F}_{\chi_2}\} = 1 - max\{\mathbb{F}_{\chi_1}, \mathbb{F}_{\chi_2}\}$.

Definition 10 (Biswas, 2006). A VS χ_1 of $\mathbb{X}$, for all $\varrho \in \mathbb{X}$. Then
(i) $\mathbb{T}_{\chi_1}(\varrho) = 0$ and $\mathbb{F}_{\chi_1}(\varrho) = 1$ is called zero VS of $\mathbb{X}$.
(ii) $\mathbb{T}_{\chi_1}(\varrho) = 1$ and $\mathbb{F}_{\chi_1}(\varrho) = 0$ is called unit VS of $\mathbb{X}$.

3. Basic operations

In this section, some new logarithmic square root operations are introduced for square root neutrosophic vague numbers (LSRNSVNs) and some aggregate operators are defined. Based on these basis characteristics, we can further explore linguistic developments, fuzzy numbers, and distance measures. An important fundamental operation of the LSRNSVN is defined.

Definition 11. The LSRNSVS $\overline{\chi}$ in $\mathbb{X}$ is $\overline{\chi} = \left\{ \varrho, \left\langle \left[\inf \Delta^{\mathbb{T}}_{\chi}, \sup(1 - \Delta^{\mathbb{F}}_{\chi})\right], \left[\inf \Delta^{\mathbb{I}}_{\chi}, \sup \Delta^{\mathbb{I}}_{\chi}\right], \left[\inf \Delta^{\mathbb{F}}_{\chi}, \sup(1 - \Delta^{\mathbb{T}}_{\chi})\right] \right\rangle \middle| \varrho \in \mathbb{X} \right\}$, where $\overline{\Delta^{T}_{\chi}} : \mathbb{X} \to Int([0,1])$, $\overline{\Delta^{I}_{\chi}} : \mathbb{X} \to Int([0,1])$ and $\overline{\Delta^{F}_{\chi}} : \mathbb{X} \to INT([0,1])$ are denotes the TD, ID and FD of $\varrho \in \mathbb{X}$ to $\overline{\chi}$, respectively, it is observe that $0 \le (\log_{\Upsilon} \sup(1 - \Delta^{\mathbb{F}}_{\chi})(\varrho))^2 + \sqrt{\log_{\Upsilon} \sup \Delta^{\mathbb{I}}_{\chi}(\varrho)} + \sqrt{\log_{\Upsilon} \sup(1 - \Delta^{\mathbb{T}}_{\chi})(\varrho)} \le 2$, where $\Upsilon = \prod \left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi} \right)$. For $\overline{\chi} = \left\langle \left[\inf \Delta^{\mathbb{T}}_{\chi}, \sup(1 - \Delta^{\mathbb{F}}_{\chi})\right], \left[\inf \Delta^{\mathbb{I}}_{\chi}, \sup \Delta^{\mathbb{I}}_{\chi}\right], \left[\inf \Delta^{\mathbb{F}}_{\chi}, \sup(1 - \Delta^{\mathbb{T}}_{\chi})\right] \right\rangle$ is called a logarithmic square root neutrosophic vague number (LSRNSVN).

Definition 12. For any LSRNSVN $\overline{\chi} = \left\langle \left[\inf \Delta^{\mathbb{T}}_{\chi}, \sup(1 - \Delta^{\mathbb{F}}_{\chi})\right], \left[\inf \Delta^{\mathbb{I}}_{\chi}, \sup \Delta^{\mathbb{I}}_{\chi}\right], \left[\inf \Delta^{\mathbb{F}}_{\chi}, \sup(1 - \Delta^{\mathbb{T}}_{\chi})\right] \right\rangle$, the score function of $\overline{\chi}$ is $S(\overline{\chi}) = \frac{\tau}{2}\left(\frac{X}{2} - \frac{Y}{2} + 1 - \frac{Z}{2} \right)$, where $X = (\log_{\Upsilon} \inf \Delta^{\mathbb{T}})^2 + (\log_{\Upsilon} \sup(1 - \Delta^{\mathbb{F}}))^2$, $Y = \sqrt{\log_{\Upsilon} \inf \Delta^{\mathbb{I}}} + \sqrt{\log_{\Upsilon} \sup \Delta^{\mathbb{I}}}$, $Z = \sqrt{\log_{\Upsilon} \inf \Delta^{\mathbb{F}}} + \sqrt{\log_{\Upsilon} \sup(1 - \Delta^{\mathbb{T}})}$ and $S(\overline{\chi}) \in [-1,1]$, where $\Upsilon = \prod \left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi} \right)$.

Definition 13. Let $\overrightarrow{\chi} = \left\langle [\inf \Delta^{\mathbb{T}}, \sup(1-\Delta^{\mathbb{F}})], [\inf \Delta^{\mathbb{I}}, \sup \Delta^{\mathbb{I}}], [\inf \Delta^{\mathbb{F}}, \sup(1-\Delta^{\mathbb{T}})]\right\rangle$ is a square root neutrosophic vague number (LSRNSVN). The TD, ID and FD are defined as $\left[\inf \Delta^{\mathbb{T}}, \sup(1-\Delta^{\mathbb{F}})\right] = \left[\log_{\Upsilon} \inf \Delta^{\mathbb{T}}, \log_{\Upsilon} \sup(1-\Delta^{\mathbb{F}})\right]$, $\left[\inf \Delta^{\mathbb{I}}, \sup \Delta^{\mathbb{I}}\right] = \left[\log_{\Upsilon} \inf \Delta^{\mathbb{I}}, \log_{\Upsilon} \sup \Delta^{\mathbb{I}}\right]$ and $\left[\inf \Delta^{\mathbb{F}}, \sup(1-\Delta^{\mathbb{T}})\right] = \left[1-\left(1-\log_{\Upsilon} \inf \Delta^{\mathbb{F}}\right), 1-\left(1-\log_{\Upsilon} \sup(1-\Delta^{\mathbb{T}})\right)\right]$, $\varrho \in X$ respectively, where X is a non-empty set and $\left[\inf \Delta^{\mathbb{T}}, \sup(1-\Delta^{\mathbb{F}})\right]$, $\left[\inf \Delta^{\mathbb{I}}, \sup \Delta^{\mathbb{I}}\right]$, $\left[\inf \Delta^{\mathbb{F}}, \sup(1-\Delta^{\mathbb{T}})\right] \in ([0,1])$ and $0 \leq \left(\log_{\Upsilon} \sup(1-\Delta^{\mathbb{F}})(\varrho)\right)^2 + \sqrt{\log_{\Upsilon} \sup \Delta^{\mathbb{I}}(\varrho)} + \sqrt{\log_{\Upsilon} \sup(1-\Delta^{\mathbb{T}})(\varrho)} \leq 2$, where $\Upsilon = \prod\left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$.

Definition 14. Let $\overrightarrow{\chi} = \left\langle [\inf \Delta^{\mathbb{T}}, \sup(1-\Delta^{\mathbb{F}})], [\inf \Delta^{\mathbb{I}}, \sup \Delta^{\mathbb{I}}], [\inf \Delta^{\mathbb{F}}, \sup(1-\Delta^{\mathbb{T}})]\right\rangle$, $\overrightarrow{\chi}_1 = \langle [\inf \Delta^{\mathbb{T}}_1, \sup(1-\Delta^{\mathbb{F}}_1)], [\inf \Delta^{\mathbb{I}}_1, \sup \Delta^{\mathbb{I}}_1], [\inf \Delta^{\mathbb{F}}_1, \sup(1-\Delta^{\mathbb{T}}_1)]\rangle$ and $\overrightarrow{\chi}_2 = \left\langle [\inf \Delta^{\mathbb{T}}_2, \sup(1-\Delta^{\mathbb{F}}_2)], [\inf \Delta^{\mathbb{I}}_2, \sup \Delta^{\mathbb{I}}_2], [\inf \Delta^{\mathbb{F}}_2, \sup(1-\Delta^{\mathbb{T}}_2)]\right\rangle$ be the any three LSRNSVNs, and $\Gamma > 0$ and $\Upsilon = \prod\left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$. Then,

1. $\overrightarrow{\chi}_1 \bigwedge \overrightarrow{\chi}_2 =$

$$\left[\begin{array}{c}\left[\begin{array}{c}\left(\sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_1} + \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_2} - \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_1} \cdot \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_2}\right)^{2\Gamma}, \\ \left(\sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_1)} + \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_2)} - \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_1)} \cdot \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_2)}\right)^{2\Gamma}\end{array}\right], \\ \left[\begin{array}{c}\left(\sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1} + \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_2} - \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1} \cdot \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_2}\right)^{\Gamma}, \\ \left(\sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1} + \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_2} - \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1} \cdot \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_2}\right)^{\Gamma}\end{array}\right], \\ \left[\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_1 \cdot \log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_2, \log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_1) \cdot \log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_2)\right]\end{array}\right],$$

2. $\overrightarrow{\chi}_1 \bigvee \overrightarrow{\chi}_2 =$

$$\left[\begin{array}{c}\left[\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_1 \cdot \log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_2, \log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_1) \cdot \log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_2)\right], \\ \left[\begin{array}{c}\left(\sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1} + \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_2} - \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1} \cdot \sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_2}\right)^{\Gamma}, \\ \left(\sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1} + \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_2} - \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1} \cdot \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_2}\right)^{\Gamma}\end{array}\right], \\ \left[\begin{array}{c}\left(\sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_1} + \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_2} - \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_1} \cdot \sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_2}\right)^{2\Gamma}, \\ \left(\sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_1)} + \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_2)} - \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_1)} \cdot \sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_2)}\right)^{2\Gamma}\end{array}\right]\end{array}\right]$$

3. $$\Gamma \cdot \overrightarrow{\chi} = \left[\begin{array}{c}\left[\left(1-\left(1-\sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}}\right)^{\Gamma}\right)^{2\Gamma}, \left(1-\left(1-\sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right], \\ \left[\sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1}, \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1}\right], \left[(\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}})^{\Gamma}, (\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}))^{\Gamma}\right]\end{array}\right],$$

4. $$\overrightarrow{\chi}^{\Gamma} = \left[\begin{array}{c}\left[(\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}})^{\Gamma}, (\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}))^{\Gamma}\right], \left[\sqrt[\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1}, \sqrt[\Gamma]{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1}\right], \\ \left[\left(1-\left(1-\sqrt[2\Gamma]{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}}\right)^{\Gamma}\right)^{2\Gamma}, \left(1-\left(1-\sqrt[2\Gamma]{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}})}\right)^{\Gamma}\right)^{2\Gamma}\right]\end{array}\right].$$

4. Different distance for LSRNSVN

ED and HD are used to calculate the differences between two elements, two sets, etc. For example, they can calculate the distance between FSs, IVFSs, IFSs, interval-valued IFSs, and VSs. A few mathematical properties of the LSRNSVNs were introduced, as well as the ED and HD measures.

Definition 15. For any two LSRNSVNs $\overrightarrow{\chi}_1 = \langle [\inf \Delta^{\mathbb{T}}_1, \sup(1-\Delta^{\mathbb{F}}_1)], [\inf \Delta^{\mathbb{I}}_1, \sup \Delta^{\mathbb{I}}_1], [\inf \Delta^{\mathbb{F}}_1, \sup(1-\Delta^{\mathbb{T}}_1)]\rangle$ and $\overrightarrow{\chi}_2 = \left\langle [\inf \Delta^{\mathbb{T}}_2, \sup(1-\Delta^{\mathbb{F}}_2)], [\inf \Delta^{\mathbb{I}}_2, \sup \Delta^{\mathbb{I}}_2], [\inf \Delta^{\mathbb{F}}_2, \sup(1-\Delta^{\mathbb{T}}_2)]\right\rangle$. Then

$$\mathbb{D}_E\left(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2\right) = \frac{1}{2}\sqrt{\begin{array}{c}\left[\frac{2+X_1-Y_1-Z_1}{4} - \frac{2+X_2-Y_2-Z_2}{4}\right]^2 \\ +\frac{1}{2}\left[\frac{2+X_1-Y_1-Z_1}{4} - \frac{2+X_2-Y_2-Z_2}{4}\right]^2\end{array}}$$

and

$$\mathbb{D}_H\left(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2\right) = \frac{1}{2}\left[\begin{array}{c}\left|\frac{2+X_1-Y_1-Z_1}{4} - \frac{2+X_2-Y_2-Z_2}{4}\right| \\ +\frac{1}{2}\left|\frac{2+X_1-Y_1-Z_1}{4} - \frac{2+X_2-Y_2-Z_2}{4}\right|\end{array}\right]$$

where $\Upsilon = \prod\left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$ and $X_1 = (\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_1)^2 + (\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_1))^2$, $Y_1 = \sqrt{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_1} + \sqrt{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_1}$, $Z_1 = \sqrt{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_1} + \sqrt{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_1)}$, $X_2 = (\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_2)^2 + (\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_2))^2$, $Y_2 = \sqrt{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_2} + \sqrt{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_2}$ and $Z_2 = \sqrt{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_2} + \sqrt{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_2)}$.

Since $\mathbb{D}_E\left(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2\right)$ and $\mathbb{D}_E\left(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2\right)$ are represents the ED and HD between $\overrightarrow{\chi}_1$ and $\overrightarrow{\chi}_2$, respectively.

Theorem 1. *If any three LSRNSVNs* $\overrightarrow{\chi}_1 = \langle [\inf \Delta^{\mathbb{T}}_1, \sup(1-\Delta^{\mathbb{F}}_1)], [\inf \Delta^{\mathbb{I}}_1, \sup \Delta^{\mathbb{I}}_1], [\inf \Delta^{\mathbb{F}}_1, \sup(1-\Delta^{\mathbb{T}}_1)]\rangle$, $\overrightarrow{\chi}_2 = \left\langle [\inf \Delta^{\mathbb{T}}_2, \sup(1-\Delta^{\mathbb{F}}_2)], [\inf \Delta^{\mathbb{I}}_2, \sup \Delta^{\mathbb{I}}_2], [\inf \Delta^{\mathbb{F}}_2, \sup(1-\Delta^{\mathbb{T}}_2)]\right\rangle$, $\overrightarrow{\chi_3} = \left\langle [\inf \Delta^{\mathbb{T}}_3, \sup(1-\Delta^{\mathbb{F}}_3)], [\inf \Delta^{\mathbb{I}}_3, \sup \Delta^{\mathbb{I}}_3], [\inf \Delta^{\mathbb{F}}_3, \sup(1-\Delta^{\mathbb{T}}_3)]\right\rangle$, *then* $\mathbb{D}_E(\chi_1, \chi_2)$ *satisfies the following properties are holds.*

1. $\mathbb{D}_E(\overline{\chi}_1,\overline{\chi}_2)$ *is zero iff* $\overline{\chi}_1=\overline{\chi}_2$.
2. $\mathbb{D}_E(\overline{\chi}_1,\overline{\chi}_2)$ *and* $\mathbb{D}_E(\overline{\chi}_2,\overline{\chi}_1)$ *are co-occur.*
3. $\mathbb{D}_E(\overline{\chi}_1,\overline{\chi}_3)\le\mathbb{D}_E(\overline{\chi}_1,\overline{\chi}_2)+\mathbb{D}_E(\overline{\chi}_2,\overline{\chi}_3)$.

Proof. The proof of Theorem 1 provided in appendix. □

Corollary 2. *If any three LSRNSVNs* $\overline{\chi}_1=\langle[\inf\Delta_1^{\mathbb{T}},\sup(1-\Delta_1^{\mathbb{F}})],[\inf\Delta_1^{\mathbb{I}},\sup\Delta_1^{\mathbb{I}}],\ [\inf\Delta_1^{\mathbb{F}},\sup(1-\Delta_1^{\mathbb{T}})]\rangle,\overline{\chi}_2=\Big\langle[\inf\Delta_2^{\mathbb{T}},\sup(1-\Delta_2^{\mathbb{F}})],[\inf\Delta_2^{\mathbb{I}},\sup\Delta_2^{\mathbb{I}}],[\inf\Delta_2^{\mathbb{F}},\sup(1-\Delta_2^{\mathbb{T}})]\Big\rangle$, $\overline{\chi}_3=\Big\langle[\inf\Delta_3^{\mathbb{T}},\sup(1-\Delta_3^{\mathbb{F}})],[\inf\Delta_3^{\mathbb{I}},\sup\Delta_3^{\mathbb{I}}],[\inf\Delta_3^{\mathbb{F}},\sup(1-\Delta_3^{\mathbb{T}})]\Big\rangle$, *then* $\mathbb{D}_H(\chi_1,\chi_2)$ *satisfies the following conditions.*

1. $\mathbb{D}_H(\overline{\chi}_1,\overline{\chi}_2)$ *is zero iff* $\overline{\chi}_1=\overline{\chi}_2$.
2. $\mathbb{D}_H(\overline{\chi}_1,\overline{\chi}_2)=\mathbb{D}_H(\overline{\chi}_2,\overline{\chi}_1)$.
3. $\mathbb{D}_H(\overline{\chi}_1,\overline{\chi}_3)\le\mathbb{D}_H(\overline{\chi}_1,\overline{\chi}_2)+\mathbb{D}_H(\overline{\chi}_2,\overline{\chi}_3)$.

5. Types of aggregation operators

This section examines the advantages of aggregating square root vague sets using logarithmic AOs. Using the square root vague set with the logarithmic toolset provides a broader modeling methodology for complex phenomena. Adding these operators to the averaging and geometric operators gives the decision-maker a more comprehensive range of interpretation possibilities. A brief description of some of the families of logarithmic square root neutrosophic vague weighted averaging (LSRNSVWA), logarithmic square root neutrosophic vague weighted geometric (LSRNSVWG), logarithmic square root generalized neutrosophic vague weighted averaging (LSRGNSVWA), logarithmic square root generalized neutrosophic vague weighted geometric (LSRGNSVWG) operators are provided in this section. The main properties of the logarithmic square root AO are commutativity, idempotency, boundedness, associativity and monotonicity.

5.1. LSRNSVWA operator

Definition 16. Let $\overline{\chi}_i=\Big\langle[\inf\Delta_i^{\mathbb{T}},\sup(1-\Delta_i^{\mathbb{F}})],[\inf\Delta_i^{\mathbb{I}},\sup\Delta_i^{\mathbb{I}}],[\inf\Delta_i^{\mathbb{F}},\sup(1-\Delta_i^{\mathbb{T}})]\Big\rangle$ be the LSRNSVNs, $W=(\xi_1,\xi_2,\ldots,\xi_n)$ be the weight of $\overline{\chi}_i$, $\xi_i\ge 0$ and $\bigwedge_{i=1}^{n}\xi_i=1$. Then LSRNSVWA $(\overline{\chi}_1,\overline{\chi}_2,\ldots,\overline{\chi}_n)=\bigwedge_{i=1}^{n}\xi_i\overline{\chi}_i$.

Theorem 3. *Let* $\overline{\chi}_i=\Big\langle[\inf\Delta_i^{\mathbb{T}},\sup(1-\Delta_i^{\mathbb{F}})],[\inf\Delta_i^{\mathbb{I}},\sup\Delta_i^{\mathbb{I}}],[\inf\Delta_i^{\mathbb{F}},\sup(1-\Delta_i^{\mathbb{T}})]\Big\rangle$ *be the LSRNSVNs. Then* $LSRNSVWA(\overline{\chi}_1,\overline{\chi}_2,\ldots,\overline{\chi}_n)$=

$$\left[\begin{array}{c}\left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\inf\Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma},\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\sup(1-\Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}\right],\\ \left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{\Upsilon_i}\inf\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma},\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{\Upsilon_i}\sup\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}\right],\\ \left[\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\inf\Delta_i^{\mathbb{F}})^{\xi_i},\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\sup(1-\Delta_i^{\mathbb{T}}))^{\xi_i}\right]\end{array}\right].$$

Proof. The proof of Theorem 3 provided in appendix. □

Theorem 4. *If all* $\overline{\chi}_i=\Big\langle[\inf\Delta_i^{\mathbb{T}},\sup(1-\Delta_i^{\mathbb{F}})],[\inf\Delta_i^{\mathbb{I}},\sup\Delta_i^{\mathbb{I}}][\inf\Delta_i^{\mathbb{F}},\sup(1-\Delta_i^{\mathbb{T}})]\Big\rangle(i=1,2,\ldots,n)$ *are equal, then LSRNSVWA*$(\overline{\chi}_1,\overline{\chi}_2,\ldots,\overline{\chi}_n)=\overline{\chi}$ *(idempotency property).*

Proof. The proof of Theorem 4 provided in appendix. □

Theorem 5. *Let* $\overline{\chi}_i=\Big\langle[\inf\Delta_{ij}^{\mathbb{T}},\sup(1-\Delta_{ij}^{\mathbb{F}})],[\inf\Delta_{ij}^{\mathbb{I}},\sup\Delta_{ij}^{\mathbb{I}}][\inf\Delta_{ij}^{\mathbb{F}},\sup(1-\Delta_{ij}^{\mathbb{T}})]\Big\rangle(i=1,2,\ldots,n);(j=1,2,\ldots,i_j)$ *be the LSRNSVWA, where* $\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{T}}}=\inf\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{T}}$, $\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{T}}}=\sup\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{T}}$, $\underbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{F}})}=\inf\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{F}})$, $\overbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{F}})}=\sup\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{F}})$, $\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{I}}}=\inf\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{I}}$, $\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{I}}}=\sup\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{I}}$, $\underbrace{\log_{\Upsilon_i}\sup\Delta^{\mathbb{I}}}=\inf\log_{\Upsilon_i}\sup\Delta_{ij}^{\mathbb{I}}$, $\overbrace{\log_{\Upsilon_i}\sup\Delta^{\mathbb{I}}}=\sup\log_{\Upsilon_i}\sup\Delta_{ij}^{\mathbb{I}}$, $\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{F}}}=\inf\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{F}}$, $\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{F}}}=\sup\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{F}}$, $\underbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{T}})}=\inf\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{T}})$, $\overbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{T}})}=\sup\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{T}})$.

Then, $\Big\langle[\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{T}}},\underbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{F}})}],[\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{I}}},\underbrace{\log_{\Upsilon_i}\sup\Delta^{\mathbb{I}}}],[\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{F}}},\overbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{T}})}]\Big\rangle$

$\le LSRNSVWA(\overline{\chi}_1,\overline{\chi}_2,\ldots,\overline{\chi}_n)\le\langle[\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{T}}},\overbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{F}})}],[\overbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{I}}},\overbrace{\log_{\Upsilon_i}\sup\Delta^{\mathbb{I}}}],\ [\underbrace{\log_{\Upsilon_i}\inf\Delta^{\mathbb{F}}},\underbrace{\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{T}})}]\rangle$, *where* $1\le i\le n$, $j=1,2,\ldots,i_j$ *(boundedness property).*

Proof. The proof of Theorem 5 provided in appendix. □

Theorem 6. *Let* $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_{t_{ij}}, \sup(1-\Delta^{\mathbb{F}}_{t_{ij}})], [\inf \Delta^{\mathbb{I}}_{t_{ij}}, \sup \Delta^{\mathbb{I}}_{t_{ij}}], [\inf \Delta^{\mathbb{F}}_{t_{ij}}, \sup(1-\Delta^{\mathbb{T}}_{t_{ij}})] \right\rangle$ *and* $\overline{W}_i = \left\langle [\inf \Delta^{\mathbb{T}}_{h_{ij}}, \sup(1-\Delta^{\mathbb{F}}_{h_{ij}})], [\inf \Delta^{\mathbb{I}}_{h_{ij}}, \sup \Delta^{\mathbb{I}}_{h_{ij}}], [\inf \Delta^{\mathbb{F}}_{h_{ij}}, \sup(1-\Delta^{\mathbb{T}}_{h_{ij}})] \right\rangle$ *be the LSRNSVWAs. For any* i*, if there is* $\sqrt{\left(\log_{Y_i} \inf \Delta^{\mathbb{T}}_{t_{ij}}\right)} + \sqrt{\left(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{t_{ij}})\right)} \leq \sqrt{\left(\log_{Y_i} \inf \Delta^{\mathbb{T}}_{h_{ij}}\right)} + \sqrt{\left(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{h_{ij}})\right)}$ *and* $\sqrt{\left(\log_{Y_i} \inf \Delta^{\mathbb{I}}_{t_{ij}}\right)} + \sqrt{\left(\log_{Y_i} \sup \Delta^{\mathbb{I}}_{t_{ij}}\right)} \leq \sqrt{\left(\log_{Y_i} \inf \Delta^{\mathbb{I}}_{h_{ij}}\right)} + \sqrt{\left(\log_{Y_i} \sup \Delta^{\mathbb{I}}_{h_{ij}}\right)}$ *and* $\left(\log_{Y_i} \inf \Delta^{\mathbb{F}}_{t_{ij}}\right) + \left(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{t_{ij}})\right) \geq \left(\log_{Y_i} \inf \Delta^{\mathbb{F}}_{h_{ij}}\right) + \left(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{h_{ij}})\right)$ *or* $\overline{\chi}_i \leq \overline{W}_i$*, then* $LSRNSVWA\left(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n\right) \leq LSRNSVWA\left(\overline{W}_1, \overline{W}_2, \ldots, \overline{W}_n\right)$*, where* $(i = 1,2,\ldots,n); (j = 1,2,\ldots,i_j)$ *(monotonicity property).*

Proof. The proof of Theorem 6 provided in appendix. □

5.2. LSRNSVWG operator

In this subsection, we have studied LSRNSVWG operator and its properties.

Definition 17. Let $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{I}}_i, \sup \Delta^{\mathbb{I}}_i], [\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle$ be the LSRNSVNs. Then LSRNSVWG $(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n) = \bigvee_{i=1}^{n} \overline{\chi}_i^{\xi_i}$.

Theorem 7. *Let* $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle$ *be the LSRNSVNs. Then*

$$\begin{bmatrix} \left[\bigvee_{i=1}^{n}(\log_{Y_i} \inf \Delta^{\mathbb{T}}_i)^{\xi_i}, \bigvee_{i=1}^{n}(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_i))^{\xi_i}\right], \\ \left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{I}}_i)}\right)^{\xi_i}\right)^{\Gamma}, \left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i} \sup \Delta^{\mathbb{I}}_i)}\right)^{\xi_i}\right)^{\Gamma}\right], \\ \left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{F}}_i)}\right)^{\xi_i}\right)^{2\Gamma}, \left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_i))}\right)^{\xi_i}\right)^{2\Gamma}\right] \end{bmatrix}.$$

Proof. Theorem 3 leads to the proof.

Theorem 8. *If all* $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{I}}_i, \sup \Delta^{\mathbb{I}}_i][\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle (i = 1,2,\ldots,n)$ *are equal, then* $LSRNSVWG(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n) = \overline{\chi}$.

Proof. Theorem 4 leads to the proof.

Remark 1. Boundedness and monotonicity are guaranteed by the LSRNSVWG operator.

Proof. Theorems 5 and 6 leads to the proof.

5.3. LSRGNSVWA operator

Definition 18. Let $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{I}}_i, \sup \Delta^{\mathbb{I}}_i], [\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle$ be the LSRNSVN. Then LSRGNSVWA $(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n) = \left(\bigwedge_{i=1}^{n} \xi_i \overline{\chi}_i^{\Gamma}\right)^{1/\Gamma}$.

Theorem 9. *Let* $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{I}}_i, \sup \Delta^{\mathbb{I}}_i], [\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle$ *be the LSRNSVNs. Then LSRGNSVWA* $(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n) =$

$$\begin{bmatrix} \left[\left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{T}}_i)^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right)^{\Gamma}, \left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_i))^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right)^{\Gamma}\right], \\ \left[\left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{I}}_i)^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right)^{\Gamma}, \left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i} \sup \Delta^{\mathbb{I}}_i)^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right)^{\Gamma}\right], \\ \left[\left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{F}}_i)}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma}, \left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_i))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma}\right] \end{bmatrix}.$$

Proof. The proof of Theorem 9 provided in appendix. □

Remark 2. The LSRGNSVWA operator becomes the LSRNSVWA operator if $\Gamma = 1$.

Theorem 10. *If all* $\overline{\chi}_i = \left\langle [\inf \Delta^{\mathbb{T}}_i, \sup(1-\Delta^{\mathbb{F}}_i)], [\inf \Delta^{\mathbb{I}}_i, \sup \Delta^{\mathbb{I}}_i][\inf \Delta^{\mathbb{F}}_i, \sup(1-\Delta^{\mathbb{T}}_i)] \right\rangle (i = 1,2,\ldots,n)$ *are equal, then* $LSRGNSVWA(\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_n) = \overline{\chi}$.

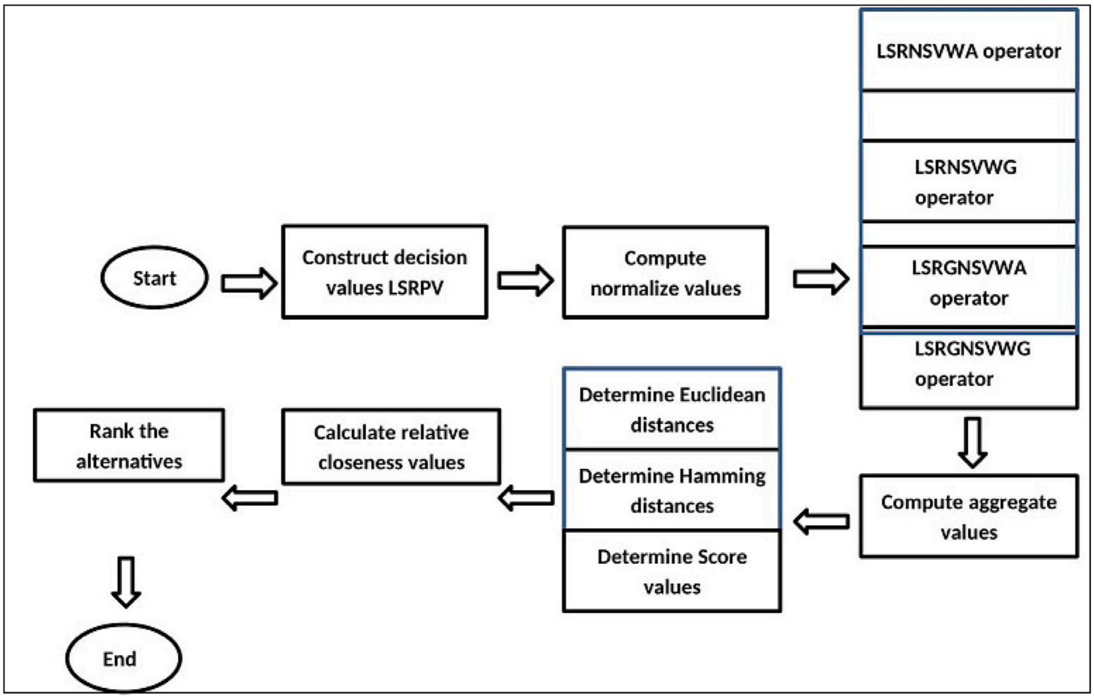

Fig. 1. Graphical representation of the algorithm.

Proof. Theorem 4 leads to the proof.

Remark 3. Boundedness and monotonicity properties of LSRGNSVWA operator are satisfied.

Proof. Theorems 5 and 6 leads to the proof.

5.4. LSRGNSVWG operator

Definition 19. Let $\overrightarrow{\chi}_i = \left\langle [\inf \Delta_i^{\mathbb{T}}, \sup(1-\Delta_i^{\mathbb{F}})], [\inf \Delta_i^{\mathbb{I}}, \sup \Delta_i^{\mathbb{I}}], [\inf \Delta_i^{\mathbb{F}}, \sup(1-\Delta_i^{\mathbb{T}})] \right\rangle$ be the LSRNSVNs. Then LSRGNSVWG $(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \dots, \overrightarrow{\chi}_n) = \frac{1}{\Gamma}\left(\bigvee_{i=1}^{n}(\Gamma\overrightarrow{\chi}_i)^{\xi_i}\right)$ $(i = 1, 2, \dots, n)$.

Theorem 11. *Let* $\overrightarrow{\chi}_i = \left\langle [\inf \Delta_i^{\mathbb{T}}, \sup(1-\Delta_i^{\mathbb{F}})], [\inf \Delta_i^{\mathbb{I}}, \sup \Delta_i^{\mathbb{I}}], [\inf \Delta_i^{\mathbb{F}}, \sup(1-\Delta_i^{\mathbb{T}})] \right\rangle$ *be the LSRNSVNs. Then LSRGNSVWG*$(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \dots, \overrightarrow{\chi}_n)$=

$$\left[\begin{array}{c} \left[\begin{array}{c} \left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{T}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma}, \\ \left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma} \end{array}\right], \\ \left[\left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{I}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right)^{\Gamma}, \left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_i^{\mathbb{I}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right)^{\Gamma}\right], \\ \left[\left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{F}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right)^{\Gamma}, \left(\left(1-\bigvee_{i=1}^{n}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right)^{\Gamma}\right] \end{array}\right].$$

Proof. Theorem 9 leads to the proof.

Remark 4. The LSRGNSVWG operator becomes the LSRNSVWG operator if $\Gamma = 1$.

Remark 5. It provides boundedness and monotonicity properties, based on the LSRGNSVWG operator.

Proof. Theorems 5 and 6 leads to the proof.

Theorem 12. *If all* $\overrightarrow{\chi}_i = \left\langle [\inf \Delta_i^{\mathbb{T}}, \sup(1-\Delta_i^{\mathbb{F}})], [\inf \Delta_i^{\mathbb{I}}, \sup \Delta_i^{\mathbb{I}}][\inf \Delta_i^{\mathbb{F}}, \sup(1-\Delta_i^{\mathbb{T}})] \right\rangle (i = 1, 2, \dots, n)$ *are equal, then LSRGNSVWG*$(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \dots, \overrightarrow{\chi}_n) = \overrightarrow{\chi}$.

6. MADM using to LSRNSV

Let $\overrightarrow{\chi} = \{\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \dots, \overrightarrow{\chi}_n\}$ be the n-alternatives, $C = \{C_1, C_2, \dots, C_m\}$ be the m-attributes, $\omega = \{\xi_1, \xi_2, \dots, \xi_m\}$ be the weights of attributes, $\overrightarrow{\chi}_{ij} = \left\langle [\inf \Delta^{\mathbb{T}}_{ij}, \sup(1-\Delta^{\mathbb{F}}_{ij})], [\inf \Delta^{\mathbb{I}}_{ij}, \sup \Delta^{\mathbb{I}}_{ij}][\inf \Delta^{\mathbb{F}}_{ij}, \sup(1-\Delta^{\mathbb{T}}_{ij})] \right\rangle$ is denote LSRNSVN of alternative $\overrightarrow{\chi}_i$ in attribute C_j.

Since $\left[\inf \Delta^{\mathbb{T}}_{ij}, \sup(1-\Delta^{\mathbb{F}}_{ij})\right], \left[\inf \Delta^{\mathbb{I}}_{ij}, \sup \Delta^{\mathbb{I}}_{ij}\right], \left[\inf \Delta^{\mathbb{F}}_{ij}, \sup(1-\Delta^{\mathbb{T}}_{ij})\right] \in [0,1]$ and $0 \preceq (\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_{ij})(\varrho))^2 + \sqrt{(\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_{ij}(\varrho))} + \sqrt{(\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_{ij})(\varrho))} \preceq 2$, where $\Upsilon = \prod \left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$. Here, n-alternative sets and m-attribute sets result in a decision matrix $n \times m$ that is indicated by the mathematical expression $\mathbb{D} = (\overrightarrow{\chi}_{ij})_{n\times m}$. Fig. 1 shows a flowchat of the algorithm for the MADM process using LSRNSV.

6.1. Algorithm

The following are the steps mentioned for solving the MADM problems.

Step-1: Form the LSRNSV choice data.

Step-2: Ascertain the decision values for normalization. Decision matrix $\mathbb{D} = (\overrightarrow{\chi}_{ij})_{n\times m}$ is normalized into $\overrightarrow{\mathbb{D}} = (\overrightarrow{\chi}_{ij})_{n\times m}$; put

$$\overrightarrow{\chi}_{ij} = \left\langle [\overrightarrow{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_{ij}}, \overrightarrow{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_{ij})}], [\overrightarrow{\log_{\Upsilon_i} \inf \Delta^{\mathbb{I}}_{ij}}, \overrightarrow{\log_{\Upsilon_i} \sup \Delta^{\mathbb{I}}_{ij}}], [\overrightarrow{\log_{\Upsilon_i} \inf \Delta^{\mathbb{F}}_{ij}}, \overrightarrow{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{T}}_{ij})}] \right\rangle$$

and $\overrightarrow{\log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_{ij}} = \log_{\Upsilon_i} \inf \Delta^{\mathbb{T}}_{ij}, \overrightarrow{\log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_{ij})} = \log_{\Upsilon_i} \sup(1-\Delta^{\mathbb{F}}_{ij})$, where $\Upsilon = \prod \left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$.

Step-3: Calculate the both ideal values for each alternative as

$$\overrightarrow{\chi}^P = \left\langle [1,1],[1,1],[0,0] \right\rangle \overrightarrow{\chi}^N = \left\langle [0,0],[0,0],[1,1] \right\rangle.$$

Step-4: Based on the two ideal values, calculate the ED between each option:

$$\mathbb{D}^P_i = \mathbb{D}_E\left(\overrightarrow{\chi}_i, \overrightarrow{\chi}^P\right); \ \mathbb{D}^N_i = \mathbb{D}_E\left(\overrightarrow{\chi}_i, \overrightarrow{\chi}^N\right).$$

Step-5: Relative closeness are calculated as $\mathbb{D}^\star_i = \frac{\mathbb{D}^N_i}{\mathbb{D}^P_i + \mathbb{D}^N_i}$.

Step-6: The output is $\sup \mathbb{D}^\star_i$. So, choosing the optimal action to take for a particular problem is a decision.

6.2. Artificial intelligence selection

Robots including service robots are becoming increasingly prevalent in society. The robots of the future will be able to manipulate objects in our daily lives with reliability, but only if they are paired with artificial intelligence techniques for planning and DM, which will allow them to comprehend how they can accomplish a particular task. AI systems are software (and perhaps also hardware) systems designed by humans to perform complex tasks by acquiring data about their environment in a physical or digital dimension in response to a complex goal. An AI system interprets the collected structured and unstructured data, interprets the knowledge, or processes the information derived from this data, and determines what action(s) to take in order to accomplish the given task. By analyzing how their decisions affect the environment, AI systems can adapt their behavior, using symbolic rules or learning numeric models. AI can be used in computer systems, which can use the knowledge it has learned to process new inputs. It is a mathematical and algorithmic skill that can only be applied to tasks that have been trained by the system. To better understand the concept of AI, picture a chatbot, whose job is to help dinners make reservations at restaurants. Developed to answer inquiries regarding table reservations, this chatbot is a computer program. By doing this, it determines how the subject is talking in general. When the chatbot has been trained, it is capable of conversing with users. As a result, the chatbot cannot assist a customer if they ask about food recommendations when they depart from the intended topic of reserving a table. Some of the existing autonomous robots can be possibly implemented in shipment delivery. Further improvements and modes of artificially intelligent last-mile delivery robots are described in the following text as considered alternatives. We offer a classification to help you decide how AI can be applied practically. Each type of AI will be briefly described, along with the most important business use cases and examples. Currently, we have selected five types of artificial intelligence robotics at random such as Analytic AI $\overrightarrow{\chi}_1$, Functional AI $\overrightarrow{\chi}_2$, Visual AI $\overrightarrow{\chi}_3$, Interactive AI $\overrightarrow{\chi}_4$ and Text AI $\overrightarrow{\chi}_5$.

1. Analytic AI $\overrightarrow{\chi}_1$: Using analytic AI, data analysis can be automated, which reduces time and labor costs. AI is increasingly capable of analyzing unstructured data sources, such as unstructured speech, images, and videos, via analytic AI tools such as NLP. AI systems have multiple advantages when it comes to analyzing data autonomously. There are many reasons for this, but the most important one is to reduce the labor cost of highly paid and highly available analytic AI professionals. Additionally, analytic AI can be used in the following ways: (i) Risk management can be improved with analytic AI, which can lead to smarter strategies and increased effectiveness. (ii) Innovative products to create new products and improve existing ones, analytic AI can analyze big data. (iii) Turbocharged supply chain is data-driven knowledge can be tapped into to solve previously unsolvable problems through analytic AI. (iv) Customer engagement is a analytic AI can be used to identify what customers want and acquire, retain and cultivate them. (v) Successful marketing campaigns is analyzing current customer purchases. The analytic AI can be used to targeted and focused campaigns.
2. Functional AI $\overrightarrow{\chi}_2$: Functional AI works in a similar manner to analytic AI in that it scans large amounts of data in order to search for patterns and dependencies between the data that it scans. Rather than giving recommendations, functional AI is designed to take action, it is not intended to make recommendations. Using its connectivity to the IoT cloud, the system is able to detect the patterns of breakdowns in a certain machine by using sensor data from the machine in question and turn the system off automatically as soon as it detects a breakdown. Secondly, we can take a look at the robots that are being used by Amazon to move shelves that contain goods to pickers, thereby speeding up the picking process.

Table 1
DM values.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\overline{\chi}_1$	$\langle$ [0.5, 0.55], [0.4, 0.45], [0.45, 0.5] $\rangle$	$\langle$ [0.5, 0.65], [0.3, 0.45], [0.35, 0.5] $\rangle$	$\langle$ [0.6, 0.65], [0.4, 0.45], [0.35, 0.4] $\rangle$	$\langle$ [0.65, 0.8], [0.25, 0.5], [0.2, 0.35] $\rangle$
$\overline{\chi}_2$	$\langle$ [0.4, 0.5], [0.6, 0.64], [0.5, 0.6] $\rangle$	$\langle$ [0.7, 0.78], [0.65, 0.7], [0.22, 0.3] $\rangle$	$\langle$ [0.5, 0.68], [0.4, 0.5], [0.32, 0.5] $\rangle$	$\langle$ [0.75, 0.85], [0.45, 0.55], [0.15, 0.25] $\rangle$
$\overline{\chi}_3$	$\langle$ [0.2, 0.21], [0.7, 0.75], [0.79, 0.8] $\rangle$	$\langle$ [0.69, 0.75], [0.5, 0.55], [0.25, 0.31] $\rangle$	$\langle$ [0.45, 0.55], [0.5, 0.55], [0.45, 0.55] $\rangle$	$\langle$ [0.85, 0.9], [0.35, 0.45], [0.1, 0.15] $\rangle$
$\overline{\chi}_4$	$\langle$ [0.45, 0.49], [0.2, 0.27], [0.51, 0.55] $\rangle$	$\langle$ [0.85, 0.97], [0.5, 0.55], [0.03, 0.15] $\rangle$	$\langle$ [0.65, 0.8], [0.35, 0.4], [0.2, 0.35] $\rangle$	$\langle$ [0.8, 0.9], [0.4, 0.5], [0.1, 0.2] $\rangle$
$\overline{\chi}_5$	$\langle$ [0.4, 0.42], [0.5, 0.55], [0.58, 0.6] $\rangle$	$\langle$ [0.7, 0.8], [0.45, 0.55], [0.2, 0.3] $\rangle$	$\langle$ [0.45, 0.75], [0.3, 0.45], [0.25, 0.55] $\rangle$	$\langle$ [0.7, 0.8], [0.5, 0.55], [0.2, 0.3] $\rangle$

Table 2
Υ values.

[0.0975, 0.1859]	[0.0120, 0.0456]	[0.0110, 0.0350]
[0.1050, 0.2254]	[0.0702, 0.1232]	[0.0053, 0.0225]
[0.0528, 0.0780]	[0.0613, 0.1021]	[0.0089, 0.0205]
[0.1989, 0.3422]	[0.0140, 0.0297]	[0.0003, 0.0058]
[0.0882, 0.2016]	[0.0338, 0.0749]	[0.0058, 0.0297]

Table 3
LSRNSVWA values.

	$LSRNSVWA$ *operator* ($\Gamma = 1$)
$\overline{\chi}_1$	$\langle$ [0.2652, 0.2723], [0.2438, 0.2535], [0.2256, 0.2327] $\rangle$
$\overline{\chi}_2$	$\langle$ [0.2712, 0.2838], [0.2336, 0.2372], [0.2147, 0.215] $\rangle$
$\overline{\chi}_3$	$\langle$ [0.2983, 0.3143], [0.2299, 0.2318], [0.1526, 0.1584] $\rangle$
$\overline{\chi}_4$	$\langle$ [0.2807, 0.3158], [0.2677, 0.2682], [0.1954, 0.213] $\rangle$
$\overline{\chi}_5$	$\langle$ [0.2641, 0.2966], [0.2463, 0.2468], [0.2076, 0.2205] $\rangle$

3. Interactive AI $\overline{\chi}_3$: Businesses can automate communication using this type of AI while still maintaining a high level of interaction. A chatbots or smart personal assistants are common example of this type of AI, which can answer pre-built questions and comprehend context of a conversation. A company can also benefit from interactive AI by improving the internal processes in the organization. A recent project that we worked on was developing a chatbot to facilitate vacation bookings for corporate clients.
4. Text AI $\overline{\chi}_4$: Text AI can allow businesses to recognize text, convert speech into text, translate texts, and generate content. This type of AI can still be used by companies even if they are not Google or Amazon. Text AI can be used, for instance, to power internal corporate knowledge bases. Using AI-powered knowledge bases, you can find relevant documents regardless of their keywords: the knowledge base can even find documents without keywords. AI can build semantic maps and recognize synonyms by using semantic search and natural language processing.
5. Visual AI $\overline{\chi}_5$: With the help of visual artificial intelligence, businesses can identify, recognize, classify, and sort objects based on their appearance and also gain insights from their images and videos. A computer system is used by insurance companies in order to estimate the amount of damage caused by damaged cars, and machines can grade apples based on their color and their size according to computer programs. This type of artificial intelligence can be seen in the fields of computer vision and augmented reality. In this example, we will show you how we developed a face recognition solution for a retailer to enhance and personalize customer service so that you are able to see for yourself how visual AI can be valuable for retailers. A further application we developed was an application for automatically inspecting the quality of the details manufactured by manufacturers, enabling them to control the quality of the products at any time.

There are four main factors to consider when selecting AIs are "operating processes" ($\mathbb{C}_1$), "costs" ($\mathbb{C}_2$), "time" ($\mathbb{C}_3$) and "externalities" ($\mathbb{C}_4$). Corresponding weights are $\omega = \{0.4, 0.3, 0.2, 0.1\}$. The objective of this process is to evaluate the options and select the best by evaluating them against the criteria.

Table 1 represents the DM values.

Table 2 represents the Υ values $\Upsilon = \prod \left(\Delta^{\mathbb{T}}_{\chi}, \Delta^{\mathbb{I}}_{\chi}, \Delta^{\mathbb{F}}_{\chi}\right)$

The Table 3 shows that the information for each choice with the LSRNSVWG operator:

Determine the optimum values of the following alternatives are $\overline{\chi}^P = \langle, 1, 1, 0 \rangle$ and $\overline{\chi}^N = \langle 0, 0, 1 \rangle$.

The following table displays the ED for each option under the positive and negative ideal values: $\mathbb{D}^P_1 = 0.3282$, $\mathbb{D}^P_2 = 0.3345$, $\mathbb{D}^P_3 = 0.3502$, $\mathbb{D}^P_4 = 0.3369$, $\mathbb{D}^P_5 = 0.3338$ and $\mathbb{D}^N_1 = 0.022$, $\mathbb{D}^N_2 = 0.0283$, $\mathbb{D}^N_3 = 0.044$, $\mathbb{D}^N_4 = 0.0307$, $\mathbb{D}^N_5 = 0.0276$.

The values of relative nearness are as follows. $\mathbb{D}^{\star}_1 = 0.0629$, $\mathbb{D}^{\star}_2 = 0.0779$, $\mathbb{D}^{\star}_3 = 0.1116$, $\mathbb{D}^{\star}_4 = 0.0836$, $\mathbb{D}^{\star}_5 = 0.0763$.

Ranking of alternatives are $\overline{\chi}_3 \succeq \overline{\chi}_4 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$.

Consequently, the Interactive AI $\overline{\chi}_3$ is best.

6.3. Comparison for proposed and some existing methods

In addition to applying fuzzy information measures to MADM, pattern recognition, clustering analysis, and picture segmentation, fuzzy information measures can also be used in pattern recognition. The results of both activities are sometimes the same. As a result, the results may vary. A fuzzy entropy metric or fuzzy knowledge metric may be used in a MADM situation to rank alternatives. To demonstrate their advantages and applicability, we compare our models with some existing ones. A comparison was conducted between several existing models and the proposed

Table 4
Different distances.

$\Gamma = 1$	$TOPSIS-Euclidean$ $distance\,(proposed)$	$TOPSIS-Hamming$ $distance\,(proposed)$
$LSRNSVWA$	$\overline{\chi}_3 \succeq \overline{\chi}_4 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$	$\overline{\chi}_3 \succeq \overline{\chi}_4 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$

Table 5
Existing different distances.

$\Gamma = 1$	$TOPSIS-Euclidean$ distance	$TOPSIS-Hamming$ distance
PNSIVWA (Palanikumar et al., 2022)	$\overline{\chi}_3 \succeq \overline{\chi}_4 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$	$\overline{\chi}_3 \succeq \overline{\chi}_4 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$
Log qVWA (Palanikumar et al., 2024)	$\overline{\chi}_1 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_4 \succeq \overline{\chi}_3$	$\overline{\chi}_1 \succeq \overline{\chi}_2 \succeq \overline{\chi}_5 \succeq \overline{\chi}_4 \succeq \overline{\chi}_3$

Table 6
LSRNSVWA values.

	$LSRNSVWA$ $operator$ $(\Gamma = 2)$
$\overline{\chi}_1$	$\langle$ [0.2646, 0.2706], [0.2426, 0.2534], [0.2256, 0.2327] $\rangle$
$\overline{\chi}_2$	$\langle$ [0.2672, 0.2782], [0.2299, 0.2346], [0.2147, 0.215] $\rangle$
$\overline{\chi}_3$	$\langle$ [0.2866, 0.298], [0.2244, 0.2266], [0.1526, 0.1584] $\rangle$
$\overline{\chi}_4$	$\langle$ [0.2723, 0.29], [0.2626, 0.2635], [0.1954, 0.213] $\rangle$
$\overline{\chi}_5$	$\langle$ [0.2606, 0.2878], [0.2442, 0.2462], [0.2076, 0.2205] $\rangle$

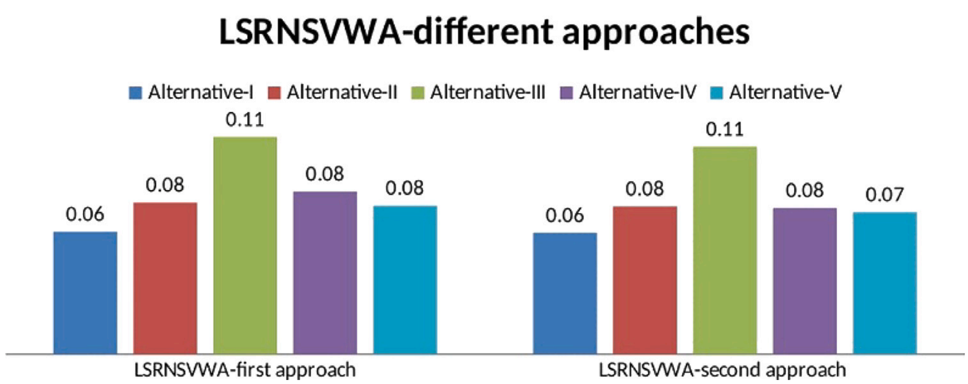

Fig. 2. EDs of different approaches using LSRNSVWA.

ones. Due to its value and advantages, it proves to be beneficial. Palanikumar et al. (2022) discusses the development of a new type of NSS with normal AOs. As a result of the facts outlined above, we utilize the LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG approaches. There are different distances as follows:

Table 4 shows that different AOs based on ED and HD. Table 5 shows that different AOs for interval valued Pythagorean normal weighted averaging, interval valued Pythagorean normal weighted geometric, generalized interval valued Pythagorean normal weighted averaging and generalized interval valued Pythagorean normal weighted geometric (Palanikumar et al., 2022).

6.4. Data analysis

MADM approaches are more reliable in certain circumstances than in others. There is a list of prerequisites for the tests. Based on different Γ values, the LSRNSVWA method calculates closeness values and rankings as follows. Adjust the $\Gamma = 2$ setting for the LSRNSVWA approaches. Orders and values of relative proximity are as follows:

Table 6 shows that for each choice with the LSRNSVWA operator: The positive and negative ideal values such as $\overline{\chi}^P = \langle [1,1],[1,1],[0,0] \rangle$ and $\overline{\chi}^N = \langle [0,0],[0,0],[1,1] \rangle$.

The following table shows the ED between every choice using the both ideal values: $\mathbb{D}_1^P = 0.328$, $\mathbb{D}_2^P = 0.3336$, $\mathbb{D}_3^P = 0.3477$, $\mathbb{D}_4^P = 0.3332$, $\mathbb{D}_5^P = 0.3323$ and $\mathbb{D}_1^N = 0.0218$, $\mathbb{D}_2^N = 0.0275$, $\mathbb{D}_3^N = 0.0415$, $\mathbb{D}_4^N = 0.027$, $\mathbb{D}_5^N = 0.0261$.

The relative nearness values are $\mathbb{D}_1^\star = 0.0624$, $\mathbb{D}_2^\star = 0.076$, $\mathbb{D}_3^\star = 0.1066$, $\mathbb{D}_4^\star = 0.0751$, $\mathbb{D}_5^\star = 0.0729$.

As can be seen from the data above, the alternate ranking is determined using the LSRNSVWA operator. If $\Gamma = 2$, the ranking of the alternatives in a new order is $\overline{\chi}_3 \succeq \overline{\chi}_2 \succeq \overline{\chi}_4 \succeq \overline{\chi}_5 \succeq \overline{\chi}_1$. As a result, $\overline{\chi}_4$ becomes the preferred option to $\overline{\chi}_2$. The basis for alternative rankings is the Γ operators LSRNSVWG, LSRGNSVWA, and LSRGNSVWG.

Fig. 2 shows different EDs using LSRNSVWA.

6.5. Advantages

This paper discusses the proposed concept in terms of its advantages and benefits. It is a generalized form of the square root aggregating operator based on VS. By reducing the square of truth membership, the root of indeterminacy membership, and the root of false membership to one, LSRNSVS becomes a square root operator based on VS. Several new square root operators are introduced, including LSRNSVWA, LSRNSVWG, LSRNSGVWA, and LSRNSGVWG. One of the main advantages of this method is the ability to objectively and subjectively assess DM procedures using several experts. A robotic selection can be characterized as an uncertain series of steps and procedures, which makes it an exciting topic for analysis. Using the illustrative example, several options and alternatives depend on the complex attitudinal characteristics of decision-makers among an ideal set of characteristics, thereby making it possible to compare the available options and choices. An essential component of this method is a square root weighted operator derived from the weighted aggregating model. In other words, it exhibits the same properties. Its characteristics are extended to consider a broader range of complex problems as the primary motivation. Among the main advantages of square root weighted operators are that they introduce distance measures, namely HDs and EDs, which can be used to compare an optimal set of preferences with alternatives or options chosen by decision-makers.

6.6. Comparison analysis

This method is effective because it considers the relationships between different attributes. Thus, the proposed method is better ranking results. Therefore, this method is more efficient than in Palanikumar et al. (2022) and Palanikumar et al. (2024). In this study, ED and HD were established for LSRNSVS. ED and HD were compared to demonstrate their superiority. We introduced a new concept of ED and HD for LSRNSVS, which were presented in a simple mathematical form. This represents an advantage in actual calculations. Consequently, a numerical example illustrated the superiority of the ED and HD when these two factors were considered. After that, all of the alternatives were ranked and the best option was chosen. As a result, Interactive AI $\overline{\chi}_3$ was the best alternative. Tables 4 and 5 compared the proposed method with the existing approaches. As discussed above, the proposed method appeared to be more general and accurate than some existing approaches. Thus, it can be used to design intelligent systems for image recognition and other real-world applications.

7. Conclusion

We presented both ED and HD for LSRNSVSs. The mathematical simplicity of these distance measures made them advantageous. A numerical example illustrated the superiority of ED and HD. It was demonstrated that both ED and HD are applicable. The rules for aggregation operations have been proposed for LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG. As well as providing some examples, we discussed some of the features of these operators. By using the LSRNSV multi-attribute decision-making technique, it was possible to select the best course of action from multiple options when uncertain and inconsistent conditions exist. The LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG operators were applied to MADM issues based on Γ. It was possible to discover the distinct ordering of alternatives by using the LSRNSVWA, LSRNSVWG, LSRGNSVWA, and LSRGNSVWG operators. As a result of the study presented above, Γ had the greatest influence over alternative rankings. By setting Γ according to the actual scenario, the decision-makers might arrive at the most reasonable ranking. A number of practical uses could be made of the ED and HD measures.

A method for handling MADM problems with uncertainty in the form of vague information was developed to demonstrate the effectiveness and consistency of the suggested square root AOs. A real-life example is provided to assess and demonstrate the applicability of our proposed method. The recommended aggregation methods were tested using the existing tools to prove their superiority and validity. The proposed AO is more reliable and accurate than the existing method. The square root AO, which we suggest for the MADM problem, is a novel method for identifying the best alternative. The aggregating operator uses distance measures, such as EDs and HDs, to consider an optimal set of preferences. Generalized aggregating operators provide a parameterized family of distance AOs useful for DM. Consequently, the results may lead to different decisions depending on the case.

The proposed model can solve several real-life problems, such as AI in education, health care, business, manufacturing, roads, machine learning, game theory, and computer science. Our proposed technique for determining the best option in MADM, the square root AO, provides a novel approach. This methodology has a wide range of potential applications. For example, could be also used in studies on green supplier selection, industrial strategies, risk assessment, predictive maintenance, as well as other innovative DM domains.

CRediT authorship contribution statement

Murugan Palanikumar: Writing – review & editing, Writing – original draft, Methodology. **Chiranjibe Jana:** Writing – review & editing, Writing – original draft, Methodology. **Ibrahim M. Hezam:** Writing – review & editing, Writing – original draft, Methodology, Data curation. **Abdelaziz Foul:** Writing – review & editing, Writing – original draft, Validation, Supervision. **Vladimir Simic:** Writing – review & editing, Writing – original draft, Methodology. **Dragan Pamucar:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix

Proof of Theorem 1. . Now, $\left(\mathbb{D}_E(\vec{\chi}_1,\vec{\chi}_2)+\mathbb{D}_E(\vec{\chi}_2,\vec{\chi}_3)\right)^2=$

$$\left[\frac{1}{2}\sqrt{\begin{array}{l}\left[\begin{array}{l}\frac{1+(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}})}}{4}\\-\frac{1+(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}})}}{4}\end{array}\right]^2\\+\frac{1}{2}\left[\begin{array}{l}\frac{1+(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}})}}{4}\\-\frac{1+(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}})}}{4}\end{array}\right]^2\end{array}}\right.$$
$$\left.+\frac{1}{2}\sqrt{\begin{array}{l}\left[\begin{array}{l}\frac{1+(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}})}}{4}\\-\frac{1+(\log_{Y_i}\inf\Delta_3^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_3^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_3^{\mathbb{T}})}}{4}\end{array}\right]^2\\+\frac{1}{2}\left[\begin{array}{l}\frac{1+(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}})}}{4}\\-\frac{1+(\log_{Y_i}\inf\Delta_3^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_3^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_3^{\mathbb{T}})}}{4}\end{array}\right]^2\end{array}}\right]^2$$

implies

$$\frac{1}{4}\left((\Xi_a-\Xi_b)^2+\frac{1}{2}(\Xi_a-\Xi_b)^2\right)+\frac{1}{4}\left((\Xi_b-\Xi_c)^2+\frac{1}{2}(\Xi_b-\Xi_c)^2\right)$$
$$+\frac{1}{2}\left(\sqrt{(\Xi_a-\Xi_b)^2+\frac{1}{2}(\Xi_a-\Xi_b)^2}\times\sqrt{(\Xi_b-\Xi_c)^2+\frac{1}{2}(\Xi_b-\Xi_c)^2}\right),$$

Since,

$$\Xi_a=\frac{1+(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_1^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_1^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}})}}{4},$$

$$\Xi_b=\frac{1+(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_2^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_2^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}})}}{4},$$

$$\Xi_c=\frac{1+(\log_{Y_i}\inf\Delta_3^{\mathbb{T}})^2-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\inf\Delta_3^{\mathbb{F}}}+1+(\log_{Y_i}\sup(1-\Delta_3^{\mathbb{F}}))^2-\sqrt{\log_{Y_i}\sup\Delta_3^{\mathbb{I}}}-\sqrt{\log_{Y_i}\sup(1-\Delta_3^{\mathbb{T}})}}{4}.$$

Hence, $\left(\mathbb{D}_E(\vec{\chi}_1,\vec{\chi}_2)+\mathbb{D}_E(\vec{\chi}_2,\vec{\chi}_3)\right)^2$

$$\begin{aligned}&\geq \frac{1}{4}\left((\Xi_a-\Xi_b)^2+\frac{1}{2}(\Xi_a-\Xi_b)^2\right)+\frac{1}{4}\left((\Xi_b-\Xi_c)^2+\frac{1}{2}(\Xi_b-\Xi_c)^2\right)\\&\quad+\frac{1}{2}\left((\Xi_a-\Xi_b)\times(\Xi_b-\Xi_c)+\frac{1}{2}(\Xi_a-\Xi_b)\times(\Xi_b-\Xi_c)\right)\\&=\frac{1}{4}\left((\Xi_a-\Xi_b)^2+(\Xi_b-\Xi_c)^2+2(\Xi_a-\Xi_b)\times(\Xi_b-\Xi_c)\right)\\&\quad+\frac{1}{4}\left(\frac{1}{2}(\Xi_a-\Xi_b)^2+\frac{1}{2}(\Xi_b-\Xi_c)^2+(\Xi_a-\Xi_b)\times(\Xi_b-\Xi_c)\right)\\&=\frac{1}{4}(\Xi_a-\Xi_b+\Xi_b-\Xi_c)^2+\frac{1}{8}(\Xi_a-\Xi_b+\Xi_b-\Xi_c)^2\\&=\frac{1}{4}(\Xi_a-\Xi_c)^2+\frac{1}{8}(\Xi_a-\Xi_c)^2\\&=\frac{1}{4}\left[(\Xi_a-\Xi_c)^2+\frac{1}{2}(\Xi_a-\Xi_c)^2\right]\\&=\mathbb{D}_E(\vec{\chi}_1,\vec{\chi}_3)^2.\end{aligned}$$

The proof of Theorem 3. It was proved using mathematical induction.

If $n=2$, then $\text{LSRNSVWA}(\vec{\chi}_1,\vec{\chi}_2)=\xi_1\vec{\chi}_1\bigwedge\xi_2\vec{\chi}_2$, where

$$\xi_1\vec{\chi}_1=\left[\begin{array}{c}\left[\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})}\right)^{\xi_1}\right)^{2\Gamma},\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))}\right)^{\xi_1}\right)^{2\Gamma}\right],\\\left[\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)^{\Gamma},\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)^{\Gamma}\right],\\\left[(\log_{Y_i}\inf\Delta_1^{\mathbb{F}})^{\xi_1},(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}}))^{\xi_1}\right]\end{array}\right]$$

$$\xi_2\overrightarrow{\chi}_2=\begin{bmatrix}\left[\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})}\right)^{\xi_2}\right)^{2\Gamma},\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))}\right)^{\xi_2}\right)^{2\Gamma}\right],\\ \left[\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)^{\Gamma},\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)^{\Gamma}\right],\\ \left[(\log_{Y_i}\inf\Delta_2^{\mathbb{F}})^{\xi_2},(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}}))^{\xi_2}\right]\end{bmatrix}.$$

Now,

$$\xi_1\overrightarrow{\chi}_1\bigwedge\xi_2\overrightarrow{\chi}_2=\begin{bmatrix}\left[\begin{array}{c}\left(\begin{array}{c}\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})}\right)^{\xi_1}\right)+\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})}\right)^{\xi_2}\right)\\ -\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})}\right)^{\xi_1}\right)\cdot\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})}\right)^{\xi_2}\right)\end{array}\right)^{2\Gamma},\\ \left(\begin{array}{c}\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))}\right)^{\xi_1}\right)+\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))}\right)^{\xi_2}\right)\\ -\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))}\right)^{\xi_1}\right)\cdot\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))}\right)^{\xi_2}\right)\end{array}\right)^{2\Gamma}\end{array}\right],\\ \left[\begin{array}{c}\left(\begin{array}{c}\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)+\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)\\ -\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)\cdot\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)\end{array}\right)^{\Gamma},\\ \left(\begin{array}{c}\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)+\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)\\ -\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\right)\cdot\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)\end{array}\right)^{\Gamma}\end{array}\right],\\ \left[(\log_{Y_i}\inf\Delta_1^{\mathbb{F}})^{\xi_1}(\log_{Y_i}\inf\Delta_2^{\mathbb{F}})^{\xi_2},(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}}))^{\xi_1}(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}}))^{\xi_2}\right]\end{bmatrix}$$

$$=\begin{bmatrix}\left[\begin{array}{c}\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{T}})}\right)^{\xi_1}\cdot\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{T}})}\right)^{\xi_2}\right)^{2\Gamma},\\ \left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))}\right)^{\xi_1}\cdot\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))}\right)^{\xi_2}\right)^{2\Gamma}\end{array}\right],\\ \left[\begin{array}{c}\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\cdot\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)^{\Gamma},\\ \left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_1^{\mathbb{I}})}\right)^{\xi_1}\cdot\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_2^{\mathbb{I}})}\right)^{\xi_2}\right)^{\Gamma}\end{array}\right],\\ \left[(\log_{Y_i}\inf\Delta_1^{\mathbb{F}})^{\xi_1}\cdot(\log_{Y_i}\inf\Delta_2^{\mathbb{F}})^{\xi_2},(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}}))^{\xi_1}\cdot(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}}))^{\xi_2}\right]\end{bmatrix}$$

$LSRNSVWA(\overrightarrow{\chi}_1,\overrightarrow{\chi}_2)=$

$$\begin{bmatrix}\left[\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma},\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}\right],\\ \left[\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma},\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}\right],\\ \left[\bigvee_{i=1}^{2}(\log_{Y_i}\inf\Delta_i^{\mathbb{F}})^{\xi_i},\bigvee_{i=1}^{2}(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))^{\xi_i}\right]\end{bmatrix}.$$

Also, valid for $n\geq 3$ and hence $LSRNSVWA(\overrightarrow{\chi}_1,\overrightarrow{\chi}_2,\dots,\overrightarrow{\chi}_m)=$

$$\begin{bmatrix}\left[\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma},\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}\right],\\ \left[\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma},\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup\Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}\right],\\ \left[\bigvee_{i=1}^{m}(\log_{Y_i}\inf\Delta_i^{\mathbb{F}})^{\xi_i},\bigvee_{i=1}^{m}(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))^{\xi_i}\right]\end{bmatrix}.$$

If $n = m + 1$, then LSRNSVWA $(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \ldots, \overrightarrow{\chi}_m, \overrightarrow{\chi}_{m+1})$

$$= \begin{bmatrix} \left[\begin{pmatrix} \bigwedge_{i=1}^{m} \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right) + \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_{m+1}^{\mathbb{T}})}\right)^{\xi_{m+1}}\right) \\ - \bigvee_{i=1}^{m} \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right) \cdot \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_{m+1}^{\mathbb{T}})}\right)^{\xi_{m+1}}\right) \end{pmatrix}^{2\Gamma}, \\ \begin{pmatrix} \bigwedge_{i=1}^{m} \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right) + \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_{m+1}^{\mathbb{F}}))}\right)^{\xi_{m+1}}\right) \\ - \bigvee_{i=1}^{m} \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right) \cdot \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_{m+1}^{\mathbb{F}}))}\right)^{\xi_{m+1}}\right) \end{pmatrix}^{2\Gamma} \right], \\ \left[\begin{pmatrix} \bigwedge_{i=1}^{m} \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right) + \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_{m+1}^{\mathbb{I}})}\right)^{\xi_{m+1}}\right) \\ - \bigvee_{i=1}^{m} \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right) \cdot \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_{m+1}^{\mathbb{I}})}\right)^{\xi_{m+1}}\right) \end{pmatrix}^{\Gamma}, \\ \begin{pmatrix} \bigwedge_{i=1}^{m} \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right) + \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_{m+1}^{\mathbb{I}})}\right)^{\xi_{m+1}}\right) \\ - \bigvee_{i=1}^{m} \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right) \cdot \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_{m+1}^{\mathbb{I}})}\right)^{\xi_{m+1}}\right) \end{pmatrix}^{\Gamma} \right], \\ \left[\bigvee_{i=1}^{m} (\log_{Y_i} \inf \Delta_i^{\mathbb{F}})^{\xi_i} \cdot (\log_{Y_i} \inf \Delta_{m+1}^{\mathbb{F}})^{\xi_{m+1}}, \bigvee_{i=1}^{m} (\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{T}}))^{\xi_i} \cdot (\log_{Y_i} \sup(1 - \Delta_{m+1}^{\mathbb{T}}))^{\xi_{m+1}} \right] \end{bmatrix}$$

$$= \begin{bmatrix} \left[\left(1 - \bigvee_{i=1}^{m+1} \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma}, \left(1 - \bigvee_{i=1}^{m+1} \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma} \right], \\ \left[\left(1 - \bigvee_{i=1}^{m+1} \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}, \left(1 - \bigvee_{i=1}^{m+1} \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma} \right], \\ \left[\bigvee_{i=1}^{m+1} (\log_{Y_i} \inf \Delta_i^{\mathbb{F}})^{\xi_i}, \bigvee_{i=1}^{m+1} (\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{T}}))^{\xi_i} \right] \end{bmatrix}.$$

The proof of Theorem 4. Given that $[\inf \Delta_i^{\mathbb{T}}, \sup(1 - \Delta_i^{\mathbb{F}})] = [\inf \Delta^{\mathbb{T}}, \sup(1 - \Delta^{\mathbb{F}})]$, $[\inf \Delta_i^{\mathbb{I}}, \sup \Delta_i^{\mathbb{I}}] = [\inf \Delta^{\mathbb{I}}, \sup \Delta^{\mathbb{I}}]$ and $[\inf \Delta_i^{\mathbb{F}}, \sup(1 - \Delta_i^{\mathbb{T}})] = [\inf \Delta^{\mathbb{F}}, \sup(1 - \Delta^{\mathbb{T}})]$, for $i = 1, 2, \ldots, n$ and $\bigwedge_{i=1}^{n} \xi_i = 1$. Now, $LSRNSVWA(\overrightarrow{\chi}_1, \overrightarrow{\chi}_2, \ldots, \overrightarrow{\chi}_n)$

$$= \begin{bmatrix} \left[\left(1 - \bigvee_{i=1}^{n} \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma}, \left(1 - \bigvee_{i=1}^{n} \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma} \right], \\ \left[\left(1 - \bigvee_{i=1}^{n} \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}, \left(1 - \bigvee_{i=1}^{n} \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma} \right], \\ \left[\bigvee_{i=1}^{n} (\log_{Y_i} \inf \Delta_i^{\mathbb{F}})^{\xi_i}, \bigvee_{i=1}^{n} (\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{T}}))^{\xi_i} \right] \end{bmatrix},$$

$$= \begin{bmatrix} \left[\left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)^{\bigwedge_{i=1}^{n} \xi_i}\right)^{2\Gamma}, \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)^{\bigwedge_{i=1}^{n} \xi_i}\right)^{2\Gamma} \right], \\ \left[\left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)^{\bigwedge_{i=1}^{n} \xi_i}\right)^{\Gamma}, \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)^{\bigwedge_{i=1}^{n} \xi_i}\right)^{\Gamma} \right], \\ \left[(\log_{Y_i} \inf \Delta_i^{\mathbb{F}})^{\bigwedge_{i=1}^{n} \xi_i}, (\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{T}}))^{\bigwedge_{i=1}^{n} \xi_i} \right] \end{bmatrix},$$

$$= \begin{bmatrix} \left[\left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{T}})}\right)\right)^{2\Gamma}, \left(1 - \left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{F}}))}\right)\right)^{2\Gamma} \right], \\ \left[\left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta_i^{\mathbb{I}})}\right)\right)^{\Gamma}, \left(1 - \left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta_i^{\mathbb{I}})}\right)\right)^{\Gamma} \right], \\ \left[(\log_{Y_i} \inf \Delta_i^{\mathbb{F}}), (\log_{Y_i} \sup(1 - \Delta_i^{\mathbb{T}})) \right] \end{bmatrix},$$

$= \overline{\chi}$.

The proof of Theorem 5. Since, $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}} = \inf \log_{Y_i} \inf \Delta^{\mathbb{T}}_{ij}$, $\overbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}} = \sup \log_{Y_i} \inf \Delta^{\mathbb{T}}_{ij}$ $\underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})} = \inf \log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{ij})$, $\overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})} = \sup \log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{ij})$ and $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}} \preceq \log_{Y_i} \inf \Delta^{\mathbb{T}}_{ij} \preceq \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}}$ and $\underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})} \preceq \log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{ij}) \preceq \overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})}$. Now, $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}} + \underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})}$

$$
\begin{aligned}
&= \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{\underbrace{(\log_{Y_i} \inf \Delta^{\mathbb{T}})}}\right)^{\xi_i}\right)^{2\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{\underbrace{(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}))}}\right)^{\xi_i}\right)^{2\Gamma} \\
&\preceq \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{T}}_{ij})}\right)^{\xi_i}\right)^{2\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}_{ij}))}\right)^{\xi_i}\right)^{2\Gamma} \\
&\preceq \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{\overbrace{(\log_{Y_i} \inf \Delta^{\mathbb{T}})}}\right)^{\xi_i}\right)^{2\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[2\Gamma]{\overbrace{(\log_{Y_i} \sup(1-\Delta^{\mathbb{F}}))}}\right)^{\xi_i}\right)^{2\Gamma} \\
&= \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{T}}} + \overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{F}})}.
\end{aligned}
$$

Since, $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}} = \inf \log_{Y_i} \inf \Delta^{\mathbb{I}}_{ij}$, $\overbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}} = \sup \log_{Y_i} \inf \Delta^{\mathbb{I}}_{ij}$ $\underbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}} = \inf \log_{Y_i} \sup \Delta^{\mathbb{I}}_{ij}$, $\overbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}} = \sup \log_{Y_i} \sup \Delta^{\mathbb{I}}_{ij}$ and $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}} \preceq \log_{Y_i} \inf \Delta^{\mathbb{I}}_{ij} \preceq \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}}$ and $\underbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}} \preceq \log_{Y_i} \sup \Delta^{\mathbb{I}}_{ij} \preceq \overbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}}$. Now, $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}} + \underbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}}$

$$
\begin{aligned}
&= \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{\underbrace{(\log_{Y_i} \inf \Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{\underbrace{(\log_{Y_i} \sup \Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma} \\
&\preceq \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{(\log_{Y_i} \inf \Delta^{\mathbb{I}}_{ij})}\right)^{\xi_i}\right)^{\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{(\log_{Y_i} \sup \Delta^{\mathbb{I}}_{ij})}\right)^{\xi_i}\right)^{\Gamma} \\
&\preceq \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{\overbrace{(\log_{Y_i} \inf \Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma} + \left(1 - \bigvee_{i=1}^{n}\left(1 - \sqrt[\Gamma]{\overbrace{(\log_{Y_i} \sup \Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma} \\
&= \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{I}}} + \overbrace{\log_{Y_i} \sup \Delta^{\mathbb{I}}}.
\end{aligned}
$$

Since, $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}} = \inf \log_{Y_i} \inf \Delta^{\mathbb{F}}_{ij}$, $\overbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}} = \sup \log_{Y_i} \inf \Delta^{\mathbb{F}}_{ij}$ $\underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})} = \inf \log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{ij})$, $\overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})} = \sup \log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{ij})$ and $\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}} \preceq \log_{Y_i} \inf \Delta^{\mathbb{F}}_{ij} \preceq \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}}$ and $\underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})} \preceq \log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{ij}) \preceq \overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})}$. Now,

$$
\begin{aligned}
\underbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}} + \underbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})} &= \bigvee_{i=1}^{n} \underbrace{(\log_{Y_i} \inf \Delta^{\mathbb{F}})}^{\xi_i} + \bigvee_{i=1}^{n} \underbrace{(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}))}^{\xi_i} \\
&\preceq \bigvee_{i=1}^{n} (\log_{Y_i} \inf \Delta^{\mathbb{F}}_{ij})^{\xi_i} + \bigvee_{i=1}^{n} (\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}_{ij}))^{\xi_i} \\
&\preceq \bigvee_{i=1}^{n} \overbrace{(\log_{Y_i} \inf \Delta^{\mathbb{F}})}^{\xi_i} + \bigvee_{i=1}^{n} \overbrace{(\log_{Y_i} \sup(1-\Delta^{\mathbb{T}}))}^{\xi_i} \\
&= \overbrace{\log_{Y_i} \inf \Delta^{\mathbb{F}}} + \overbrace{\log_{Y_i} \sup(1-\Delta^{\mathbb{T}})}.
\end{aligned}
$$

Hence,

$$\frac{\bigwedge_{i=1}^{n}\xi_i}{2}\times\left[\begin{array}{l}\dfrac{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\underbrace{(\log_{\Upsilon_i}\inf\Delta^{\mathbb{T}})}}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\underbrace{(\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{F}}))}}\right)^{\xi_i}\right)^{2\Gamma}}{2}\\ -\dfrac{\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\underbrace{(\log_{\Upsilon_i}\inf\Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma}}+\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\underbrace{(\log_{\Upsilon_i}\sup\Delta^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma}}}{2}\\ +1-\dfrac{\sqrt{\left(\bigvee_{i=1}^{n}\overbrace{(\log_{\Upsilon_i}\inf\Delta^{\mathbb{F}})}{}^{\xi_i}\right)}+\sqrt{\left(\bigvee_{i=1}^{n}\overbrace{(\log_{\Upsilon_i}\sup(1-\Delta^{\mathbb{T}}))}{}^{\xi_i}\right)}}{2}\end{array}\right]$$

$$\preceq\frac{\bigwedge_{i=1}^{n}\xi_i}{2}\times\left[\begin{array}{l}\dfrac{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}}{2}\\ -\dfrac{\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}}+\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{\Upsilon_i}\sup\Delta_{ij}^{\mathbb{I}})}\right)^{\xi_i}\right)^{\Gamma}}}{2}\\ +1-\dfrac{\sqrt{\left(\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{F}})^{\xi_i}\right)}+\sqrt{\left(\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{T}}))^{\xi_i}\right)}}{2}\end{array}\right]$$

$$\preceq\frac{\bigwedge_{i=1}^{n}\xi_i}{2}\times\left[\begin{array}{l}\dfrac{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\overbrace{(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{T}})}}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\overbrace{(\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{F}}))}}\right)^{\xi_i}\right)^{2\Gamma}}{2}\\ -\dfrac{\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\overbrace{(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma}}+\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\overbrace{(\log_{\Upsilon_i}\sup\Delta_{ij}^{\mathbb{I}})}}\right)^{\xi_i}\right)^{\Gamma}}}{2}\\ +1-\dfrac{\sqrt{\left(\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\inf\Delta_{ij}^{\mathbb{F}})^{\xi_i}\right)}+\sqrt{\left(\bigvee_{i=1}^{n}(\log_{\Upsilon_i}\sup(1-\Delta_{ij}^{\mathbb{T}}))^{\xi_i}\right)}}{2}\end{array}\right].$$

Therefore, $\left\langle[\underbrace{\inf\Delta^{\mathbb{T}}},\underbrace{\sup(1-\Delta^{\mathbb{F}})}],[\underbrace{\inf\Delta^{\mathbb{I}}},\underbrace{\sup\Delta^{\mathbb{I}}}],[\overbrace{\inf\Delta^{\mathbb{F}}},\overbrace{\sup(1-\Delta^{\mathbb{T}})}]\right\rangle\preceq LSRNSVWA(\overrightarrow{\chi}_1,\overrightarrow{\chi}_2,\dots,\overrightarrow{\chi}_n)$
$\preceq\left\langle[\overbrace{\inf\Delta^{\mathbb{T}}},\overbrace{\sup(1-\Delta^{\mathbb{F}})}],[\overbrace{\inf\Delta^{\mathbb{I}}},\overbrace{\sup\Delta^{\mathbb{I}}}],[\underbrace{\inf\Delta^{\mathbb{F}}},\underbrace{\sup(1-\Delta^{\mathbb{T}})}]\right\rangle$.

The proof of Theorem 6. For any i, $\sqrt{\left(\log_{\Upsilon_i}\inf\Delta_{t_{ij}}^{\mathbb{T}}\right)}+\sqrt{\left(\log_{\Upsilon_i}\sup(1-\Delta_{t_{ij}}^{\mathbb{F}})\right)}\preceq\sqrt{\left(\log_{\Upsilon_i}\inf\Delta_{h_{ij}}^{\mathbb{T}}\right)}+\sqrt{\left(\log_{\Upsilon_i}\sup(1-\Delta_{h_{ij}}^{\mathbb{F}})\right)}$.

Therefore, $1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{t_i}^{\mathbb{T}}\right)}+1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\sup(1-\Delta_{t_i}^{\mathbb{F}})\right)}\geq 1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{h_i}^{\mathbb{T}}\right)}+1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\sup(1-\Delta_{h_i}^{\mathbb{F}})\right)}$.
Hence,

$$\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{t_i}^{\mathbb{T}}\right)}\right)^{\xi_i}+\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\sup(1-\Delta_{t_i}^{\mathbb{F}})\right)}\right)^{\xi_i}\geq$$
$$\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{h_i}^{\mathbb{T}}\right)}\right)^{\xi_i}+\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{\left(\log_{\Upsilon_i}\sup(1-\Delta_{h_i}^{\mathbb{F}})\right)}\right)^{\xi_i}$$

and

$$\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\inf\Delta_{t_i}^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\sup(1-\Delta_{t_i}^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}\preceq$$
$$\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\inf\Delta_{h_i}^{\mathbb{T}})}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{\Upsilon_i}\sup(1-\Delta_{h_i}^{\mathbb{F}}))}\right)^{\xi_i}\right)^{2\Gamma}$$

For any i, $\sqrt{\left(\log_{\Upsilon_i}\inf\Delta_{t_{ij}}^{\mathbb{I}}\right)}+\sqrt{\left(\log_{\Upsilon_i}\sup\Delta_{t_{ij}}^{\mathbb{I}}\right)}\preceq\sqrt{\left(\log_{\Upsilon_i}\inf\Delta_{h_{ij}}^{\mathbb{I}}\right)}+\sqrt{\left(\log_{\Upsilon_i}\sup\Delta_{h_{ij}}^{\mathbb{I}}\right)}$.

Therefore, $1-\sqrt[\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{t_i}^{\mathbb{I}}\right)}+1-\sqrt[\Gamma]{\left(\log_{\Upsilon_i}\sup\Delta_{t_i}^{\mathbb{I}}\right)}\geq 1-\sqrt[\Gamma]{\left(\log_{\Upsilon_i}\inf\Delta_{h_i}^{\mathbb{I}}\right)}+1-\sqrt[\Gamma]{\left(\log_{\Upsilon_i}\sup\Delta_{h_i}^{\mathbb{I}}\right)}$.

Hence,

$$\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\left(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{t_i}\right)}\right)^{\xi_i}+\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\left(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{t_i}\right)}\right)^{\xi_i}\geq$$
$$\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\left(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{h_i}\right)}\right)^{\xi_i}+\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{\left(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{h_i}\right)2}\right)^{\xi_i}$$

and

$$\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{t_i})}\right)^{\xi_i}\right)^{\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{t_i})}\right)^{\xi_i}\right)^{\Gamma}\preceq$$
$$\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{h_i})}\right)^{\xi_i}\right)^{\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{h_i})}\right)^{\xi_i}\right)^{\Gamma}$$

.

For any i, $\left(\log_{Y_i}\inf \Delta^{\mathbb{F}}_{t_{ij}}\right)+\left(\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{t_{ij}})\right)\geq\left(\log_{Y_i}\inf \Delta^{\mathbb{F}}_{h_{ij}}\right)+\left(\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{h_{ij}})\right)$.
Therefore,

$$\text{Therefore, } 1-\frac{\left(\bigvee_{i=1}^{n}\log_{Y_i}\inf \Delta^{\mathbb{F}}_{t_{ij}}\right)+\left(\bigvee_{i=1}^{n}\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{t_{ij}})\right)}{2}\preceq$$

$$1-\frac{\left(\bigvee_{i=1}^{n}\log_{Y_i}\inf \Delta^{\mathbb{F}}_{h_{ij}}\right)+\left(\bigvee_{i=1}^{n}\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{h_{ij}})\right)}{2}.$$

Hence,

$$\frac{1}{2}\times\left[\begin{array}{l}\dfrac{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{T}}_{ti})}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta^{\mathbb{F}}_{ti}))}\right)^{\xi_i}\right)^{2\Gamma}}{2}\\ -\dfrac{\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{ti})}\right)^{\xi_i}\right)^{2\Gamma}}+\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{ti})}\right)^{\xi_i}\right)^{2\Gamma}}}{2}\\ +1-\dfrac{\sqrt{\left(\bigvee_{i=1}^{n}(\log_{Y_i}\inf \Delta^{\mathbb{F}}_{tij})\right)}+\sqrt{\left(\bigvee_{i=1}^{n}(\log_{Y_i}\sup(1-\Delta^{\mathbb{T}})_{tij})\right)}}{2}\end{array}\right]$$

$$\preceq\frac{1}{2}\times\left[\begin{array}{l}\dfrac{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{T}}_{hi})}\right)^{\xi_i}\right)^{2\Gamma}+\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta^{\mathbb{F}}_{hi}))}\right)^{\xi_i}\right)^{2\Gamma}}{2}\\ -\dfrac{\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{hi})}\right)^{\xi_i}\right)^{2\Gamma}}+\sqrt{\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{hi})}\right)^{\xi_i}\right)^{2\Gamma}}}{2}\\ +1-\dfrac{\sqrt{\left(\bigvee_{i=1}^{n}(\log_{Y_i}\inf \Delta^{\mathbb{F}}_{hij})\right)}+\sqrt{\left(\bigvee_{i=1}^{n}(\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{hij}))\right)}}{2}\end{array}\right].$$

Hence, $LSRNSVWA\,(\vec{\chi}_1,\vec{\chi}_2,\ldots,\vec{\chi}_n)\preceq LSRNSVWA\left(\vec{W}_1,\vec{W}_2,\ldots,\vec{W}_n\right)$.

The proof of Theorem 9. First to prove that, $\bigwedge_{i=1}^{n}\xi_i\vec{\chi}_i^{\Gamma}=$

$$\left[\begin{array}{c}\left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{T}}_{i})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma},\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta^{\mathbb{F}}_{i}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}\right],\\ \left[\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{I}}_{i})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma},\left(1-\bigvee_{i=1}^{n}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta^{\mathbb{I}}_{i})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}\right],\\ \left[\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta^{\mathbb{F}}_{i})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i},\bigvee_{i=1}^{n}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta^{\mathbb{T}}_{i}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}\right]\end{array}\right].$$

Put $n = 2$, $\xi_1\chi_1 \bigwedge \xi_2\chi_2 =$

$$
\left[\begin{array}{c}
\left[\begin{array}{c}
\left(\begin{array}{c}
\sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_1^{\mathbb{T}})^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} + \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_2^{\mathbb{T}})^{\Gamma}}\right)^{\xi_2}\right)^{2\Gamma}} \\
-\sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_1^{\mathbb{T}})^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} \cdot \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_2^{\mathbb{T}})^{\Gamma}}\right)^{\xi_2}\right)^{2\Gamma}}
\end{array}\right)^{2\Gamma}, \\
\left(\begin{array}{c}
\sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} + \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_2}\right)^{2\Gamma}} \\
-\sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} \cdot \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_2}\right)^{2\Gamma}}
\end{array}\right)^{2\Gamma}
\end{array}\right], \\
\left[\begin{array}{c}
\left(\begin{array}{c}
\sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_1^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} + \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_2^{\mathbb{I}})^{\Gamma}}\right)^{\xi_2}\right)^{\Gamma}} \\
-\sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_1^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} \cdot \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_2^{\mathbb{I}})^{\Gamma}}\right)^{\xi_2}\right)^{\Gamma}}
\end{array}\right)^{\Gamma}, \\
\left(\begin{array}{c}
\sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_1^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} + \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_2^{\mathbb{I}})^{\Gamma}}\right)^{\xi_2}\right)^{\Gamma}} \\
-\sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_1^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} \cdot \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_2^{\mathbb{I}})^{\Gamma}}\right)^{\xi_2}\right)^{\Gamma}}
\end{array}\right)^{\Gamma}
\end{array}\right], \\
\left[\begin{array}{c}
\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_1^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1} \cdot \left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_2^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1}, \\
\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_1^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1} \cdot \left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_2^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1}
\end{array}\right]
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
\left[\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}, \left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}\right], \\
\left[\left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}, \left(1-\bigvee_{i=1}^{2}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}\right], \\
\left[\bigvee_{i=1}^{2}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}, \bigvee_{i=1}^{2}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}\right]
\end{array}\right].
$$

In general,

$$
\left[\begin{array}{c}
\left[\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}, \left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}\right], \\
\left[\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}, \left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}\right], \\
\left[\bigvee_{i=1}^{m}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}, \bigvee_{i=1}^{m}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}\right]
\end{array}\right].
$$

If $n = m + 1$, then $\bigwedge_{i=1}^{m} \xi_i \overrightarrow{\chi}_i^{\Gamma} + \xi_{m+1}\overrightarrow{\chi}_{m+1}^{\Gamma} = \bigwedge_{i=1}^{m+1} \xi_i \overrightarrow{\chi}_i^{\Gamma}$.

Now, $\bigwedge_{i=1}^{m} \xi_i \overrightarrow{\chi}_i^{\Gamma} + \xi_{m+1}\overrightarrow{\chi}_{m+1}^{\Gamma} = \bigwedge_{i=1}^{m+1} \xi_i \overrightarrow{\chi}_i^{\Gamma} = \xi_1\overrightarrow{\chi}_1^{\Gamma} \bigwedge \xi_2 \overrightarrow{\chi}_2^{\Gamma} \bigwedge \cdots \bigwedge \xi_m\overrightarrow{\chi}_m^{\Gamma} \bigwedge \xi_{m+1}\overrightarrow{\chi}_{m+1}^{\Gamma}$

$$= \left[\begin{array}{c}
\left[\begin{array}{l}
\left(\begin{array}{l}
\sqrt[2\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}} + \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_{m+1}^{\mathbb{T}})^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} \\
-\sqrt[2\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}} \cdot \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_{m+1}^{\mathbb{T}})^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}}
\end{array}\right)^{2\Gamma}, \\
\left(\begin{array}{l}
\sqrt[2\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}} + \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_{m+1}^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}} \\
-\sqrt[2\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}} \cdot \sqrt[2\Gamma]{\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_{m+1}^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_1}\right)^{2\Gamma}}
\end{array}\right)^{2\Gamma}
\end{array}\right], \\
\left[\begin{array}{l}
\left(\begin{array}{l}
\sqrt[\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}} + \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_{m+1}^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} \\
-\sqrt[\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}} \cdot \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_{m+1}^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}}
\end{array}\right)^{\Gamma}, \\
\left(\begin{array}{l}
\sqrt[\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}} + \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_{m+1}^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}} \\
-\sqrt[\Gamma]{\left(1-\bigvee_{i=1}^{m}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}} \cdot \sqrt[\Gamma]{\left(1-\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_{m+1}^{\mathbb{I}})^{\Gamma}}\right)^{\xi_1}\right)^{\Gamma}}
\end{array}\right)^{\Gamma}
\end{array}\right], \\
\left[\begin{array}{l}
\bigvee_{i=1}^{m}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i} \cdot \left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_{m+1}^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1}, \\
\bigvee_{i=1}^{m}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i} \cdot \left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_{m+1}^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_1}
\end{array}\right]
\end{array}\right]$$

Hence,

$$\bigwedge_{i=1}^{m+1}\xi_i\overrightarrow{\chi}_i^{\Gamma} = \left[\begin{array}{c}
\left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}, \left(1-\bigvee_{i=1}^{m+1}\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)^{\xi_i}\right)^{2\Gamma}\right], \\
\left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}, \left(1-\bigvee_{i=1}^{m+1}\left(1-\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)^{\xi_i}\right)^{\Gamma}\right], \\
\left[\bigvee_{i=1}^{m+1}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}, \bigvee_{i=1}^{m+1}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}\right]
\end{array}\right].$$

Also, $\left(\bigwedge_{i=1}^{m+1}\xi_i\overrightarrow{\chi}_i^{\Gamma}\right)^{1/\Gamma} =$

$$\left[\begin{array}{c}
\left[\left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{T}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right]^{\Gamma}, \left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\left(\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{F}}))^{\Gamma}}\right)\right)^{\xi_i}\right)^{2\Gamma}\right]^{\Gamma}\right], \\
\left[\left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{I}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right]^{\Gamma}, \left[\left(1-\bigvee_{i=1}^{m+1}\left(1-\left(\sqrt[\Gamma]{(\log_{Y_i}\sup \Delta_i^{\mathbb{I}})^{\Gamma}}\right)\right)^{\xi_i}\right)^{\Gamma}\right]^{\Gamma}\right], \\
\left[\begin{array}{l}
\left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{m+1}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\inf \Delta_i^{\mathbb{F}})}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma}, \\
\left(1-\left(1-\sqrt[2\Gamma]{\bigvee_{i=1}^{m+1}\left(\left(1-\left(1-\sqrt[2\Gamma]{(\log_{Y_i}\sup(1-\Delta_i^{\mathbb{T}}))}\right)^{\Gamma}\right)^{2\Gamma}\right)^{\xi_i}}\right)^{\Gamma}\right)^{2\Gamma}
\end{array}\right]
\end{array}\right].$$

It is valid for any m.

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