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Moments of a non-homogenous bi-variate fragmentation process using integral equations tools

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In this paper, we are interested in two-dimensional fragmentation process that describes the evolution of an object having a rectangular shape. We focus on a fragmentation process in which we break a rectangle according to a distribution that depends on its dimensions. Using the renewal theory, we provide the asymptotic of the mean and of the variance of the distribution of the total number of the sub-rectangles.

KEYWORDS
fragmentation process, integral equations, renewal theory

MSC CLASSIFICATION
37A50; 05C05; 60K05; 31A10

1 | INTRODUCTION

Random fragmentation has applications in several fields, such as biology,[1] physics,[2] and computer sciences,[3,4] for instance. The fragmentation process has been the interest of many authors since the works of Brennan and Durret[5] and Sibuya and Itoh.[4] Afterwards, Janson and Neininger[6] focused on a non-homogeneous fragmentation of an interval of length x, i.e., the fragmentation of the interval depends on x; they studied the case when the fragmentation probability is $p(x) = \mathbf{1}_{\{x\geq 1\}}$ and gave the asymptotic behavior of the total number of fragments obtained at the end of the process. More recently, in Aguech,[7] the fragmentation of an interval of length x was studied, with fragmentation probability is equal to $p(x) = 1 - e^{-x}$, and the asymptotic distribution of the total number of the obtained fragments was obtained.

In this paper, we focus on a two-dimensional fragmentation process that evolves over time as follows. We take a rectangle with dimensions x and y, and we cut, with probability $p(x, y) = 1_{\{x\geq 1,\ y\geq 1\}}$, independently and uniformly, x into d slides and y into d' slides where d and d' are two nonnegative integers. With complementary probability, we decide to let them definitively stable. The last is modelized as follows: Let

$$\mathbf{U} = (U_1\ \dots\ , U_d) \text{ and } \mathbf{V} = (V_1\ \dots\ , V_{d'}) \text{ be independent random vectors s.t. } \sum_{i=1}^{d} U_i = \sum_{j=1}^{d'} V_j = 1, a.s. \qquad (1)$$

The lengths of the sub-pieces obtained by cutting x and y are, respectively, $U_1 x, \dots, U_d x$, and $V_1 y, \dots, V_{d'} y$, and the procedure continues, independently until all the obtained rectangles become stable. By a stable rectangle, we mean that of its dimension is less than 1.

Our aim is to study the asymptotic behavior of the first and the second moments of the size of the fragmentation process denoted by N:

$$N(x, y) := \text{the total number of stable rectangles obtained at the end of the fragmentation process.} \tag{2}$$

In the one-dimensional case, Janson and Neininger[6] used the planar renewal theory to obtain the mean $m(x)$ of the size of the fragmentation tree, and they used the residue theorem to get a refined expression of $m(x)$. In the bi-dimensional case, it is very difficult to use the residue theorem; furthermore, the results about bi-renewal densities are quite rare. For this reason, we took interest in the bi-dimensional case and managed to establish a refined expansion of the bi-renewal density function in a general context. We recall that Janson and Neininger[6] considered the one-dimensional fragmentation process and showed that the total number $N_1(x)$ of the internal nodes in the fragmentation tree descending from the interval x is finite almost surely. The latter insures that the number $N(x, y)$ satisfies

$$N(x, y) \le N_1(x)N_1(y) \Rightarrow N(x, y) \text{ is also almost surely finite.} \tag{3}$$

We could be wrong in thinking that the study of our fragmentation process amounts to study the fragmentation process of intervals of lengths x and y and

$$N(x, y) = N_1(x)\, N_1(y). \tag{4}$$

As in Janson and Neininger,[6] we will be interested by the behavior of the expectations

$$\mathbb{M}(x) := \mathbb{E}[N_1(x)],\ \mathbb{M}(x, y) := \mathbb{E}[N(x,\ y)] \le \mathbb{M}(x)\mathbb{M}(y),\ \text{ as } x \text{ and } y \text{ tend to} \infty, \tag{5}$$

the last inequality obviously stems from (3). We emphasize that in our model, each rectangle, such that $x < 1 < y$, stays stable, and then, we are far from equality (4). To be convicted, observe that even if we start with x and y close to 1, and if $d = d' = 2$ and U_i, V_i, $i = 1, 2$ are uniform on $[0, 1]$, then

$$5 = \lim_{\epsilon \to 0+} \mathbb{M}(1 + \epsilon, 1 + \epsilon) \neq \lim_{\epsilon \to 0+} \mathbb{M}^2(1 + \epsilon) = 3^2.$$

The paper is organized as follows. In Section 2, we formalize the model, set the notations, and the assumptions. We show that the mean of the total number of unstable rectangles, $N(x, y)$, satisfies a bi-renewal equation that will be the key for the main results. In Section 3, we provide our main results on the asymptotic behavior of the mean and the variance of $N(x, y)$. In Theorem 4.7 in Section 4, we give the asymptotic behavior of the density function, solution of the bi-renewal equation. Section 5 is devoted to the proofs.

2 | DESCRIPTION OF THE MODEL, NOTATIONS, AND DEFINITIONS

2.1 | Description of the model

The model is described as follows:

- We start with a rectangle of dimensions x and y, and we fix d and d' two integers.
- Assume that, with probability $p(x, y) = 1_{\{x \ge 1,\ y \ge 1\}}$, we decide to cut, independently and uniformly, x into d slides and y into d' slides where d and d' are two non-negative integers. With complementary probability, we decide to let them definitively stable.
- We repeat, recursively and independently, the cutting procedure in (1) on all the sub-rectangles with new and independent copies of $(\mathbf{U}, \mathbf{V})$. Note that this fragmentation process stops almost surely after a finite number of steps and leaves a finite number of sub-rectangles all stable. To explain the latter, just consider inequality (3).

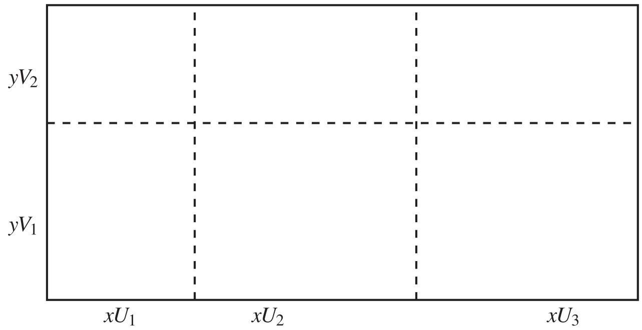

FIGURE 1 A cut rectangle at step 1 when $d = 3,\ d' = 2$

- If $x \geq 1$ and $y \geq 1$, we cut the rectangle according to the random vectors

$$\mathbf{U} = (U_1, \dots, U_d) \text{ for } x, \text{ and } \mathbf{V} = (V_1, \dots, V_{d'}) \text{ for } y.$$

- If $x < 1$ or $y < 1$, we decide to leave the rectangle definitively stable.
- We repeat independently at each step this procedure for all sub-rectangles, with independent copies of $\mathbf{U}$ and $\mathbf{V}$.

Figure 1 illustrates an example of fragmentation of a rectangle.

Note that this fragmentation process can be considered as a random tree, where the root is the first rectangle, the internal nodes are the unstable rectangles, and the leaves are the stable rectangles. Let $N(x,\ y)$ be the total number of internal nodes in the fragmentation tree, which is described as follows: In case where $x < 1$ or $y < 1$, we have $N(x, y) = 1$. We then assume that we start with a rectangle with dimensions greater than 1, and then, $N(x,\ y)$ satisfies the following equation:

$$N(x,\ y) \stackrel{d}{=} 1 + \sum_{i=1}^{d} \sum_{j=1}^{d'} N_{i,j}(xU_i,\ yV_j),$$

where, for $1 \leq i \leq d$ and $1 \leq j \leq d'$, the r.v.'s $N_{i,j}(.,.)$ are independent copies of $N(.,.)$. The results of this paper concern only the unstable sub-rectangles produced during the fragmentation process, although we can deduce immediately more data on the fragmentation tree especially with regard to stable rectangles. Indeed, as mentioned in Janson and Neininger,[6] each internal node gives birth at its death to dd' nodes which can be internal or external. The study the number of all leaves $N_l(x, y)$ follows from the fact that the total number of nodes (external and internal) is

$$N(x, y) + N_l(x, y) = 1 + dd' N(x, y).$$

2.2 | Main notations and conventions

$$\begin{aligned}
d &:= d' = 2; \\
U_2 &= 1 - U_1,\ V_2 = 1 - V_1,\ a.s. \ \text{ and } \ U_1,\ V_1 \ \text{ are independent} \\
(X_i, Y_i) &:= -\left(\ln(U_i), \ln(V_j)\right) \in [0, \infty)^2 \ i,\ j = 1, 2; \qquad (6)\\
\mu &:= \sum_{i=1}^{2} \sum_{j=1}^{2} \mathbb{P}_{X_i} \otimes \mathbb{P}_{Y_j},\ (\mathbb{P}_{X_i} \otimes \mathbb{P}_{Y_j} \text{ is the joint distribution of } (X_i, Y_j));
\end{aligned}$$

$$d\nu(s_1, s_2) := e^{-(s_1+s_2)} d\mu(s_1, s_2),\ (s_1, s_2) \in (0, \infty)^2; \tag{7}$$

$$(\theta_1, \theta_2) := \sum_{i=1}^{2} \mathbb{E}[(X_i\, e^{-X_i}, Y_i\, e^{-Y_i})] = \left(\int_{(0,\infty)^2} t_1 d\nu(t_1, t_2), \int_{(0,\infty)^2} t_2 d\nu(t_1, t_2) \right); \tag{8}$$

$$h_n := \text{p.d.f. of } \ \nu^{*n} \ \ (\text{if } \nu \text{ is absolutely continuous}); \tag{9}$$

$$\Phi(u_1, u_2) := \sum_{i=1}^{2} \mathbb{E}\left[e^{-u_1 X_i}\right] \sum_{j=1}^{2} \left[e^{-u_2 Y_j}\right], \; u_1, \; u_2 \in \mathbb{R}_+; \tag{10}$$

$$\Psi(v_1, v_2) := \int_{(0,\infty)^2} e^{i(v_1 s_1 + v_2 s_2)} d\nu(s_1, s_2) = \Phi(1 - iv_1, \; 1 - iv_2), \; (v_1, v_2) \in \mathbb{R}^2. \tag{11}$$

Note that μ is not a probability measure whereas ν is. The bi-variate function $\mathbb{M}(e^{t_1}, e^{t_2})$ satisfies the following equation:

$$\mathbb{M}(e^{t_1}, e^{t_2}) = 1 + (\mathbb{M}(e^{s_1}, e^{s_2}) * \mu)(t_1, t_2), \; (t_1, t_2) \in (0, \infty)^2.$$

The function $\mathbb{M}_e(t_1, t_2) = e^{-(t_1+t_2)}\mathbb{M}(e^{t_1}, e^{t_2})$ trivially satisfies the *bi-variate renewal equation*:

$$\mathbb{M}_e(t_1, t_2) = e^{-(t_1+t_2)} + (\mathbb{M}_e * \nu)(t_1, \; t_2). \tag{12}$$

The class $\mathcal{J}_1$ of distributions in next definition was introduced by Smith.[8] It will be useful when we need to retrieve p.d.f.'s. by Fourier inversions.

Definition 2.1. The class $\mathcal{J}_1$ is the set of distributions ω on $\mathbb{R}$ whose characteristic function

$$\Psi(v) = \int_{\mathbb{R}} e^{ivs} d\omega(s), \; v \in \mathbb{R}$$

satisfies

$$|\Psi(v)| \leq \frac{c}{|v|^{\alpha}}, \; \text{if } |v| > M, \tag{13}$$

for some non-negative numbers $\alpha, \; c$.

2.3 | Assumptions

The following assumptions are crucial in our work:

1. Each of X_1 and Y_1 has a continuous distribution.
2. The marginal distributions of the probability measure ν, given in (7), belongs to the set $\mathcal{J}_1$.
3. The p.d.f. h_n in (9) satisfies

$$\lim_{t_1 \to +\infty} t_1^{\frac{3}{2}} h_n(t_1, t_2) = \lim_{t_2 \to +\infty} t_2^{\frac{3}{2}} h_n(t_1, t_2) = 0, \; \text{for all } \; n \geq 1.$$

Example 2.2. If ν is the bi-variate exponential distribution with p.d.f. $e^{-(t_1+t_2)}$, $t_1, \; t_2 > 0$, then

$$h_n(t_1, t_2) = \frac{(t_1 t_2)^{n-1}}{(n-1)!^2} \, e^{-(t_1+t_2)},$$

and condition **(C)** is satisfied.

2.4 | Auxiliary notations:

These notations will be used in the paper. For a random vector $\mathbf{X} = (X_1, \; X_2)$ with distribution ω, with mean $\lambda = (\lambda_1, \lambda_2)$ and covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ such that $\mathbb{E}[\|\mathbf{X}\|^4] < \infty$, we set

$$\mathbf{X}^* = \left(\frac{X_1 - \lambda_1}{\sigma_1}, \; \frac{X_2 - \lambda_2}{\sigma_2} \right) := (X_1^*, X_2^*),$$

and

$$K(\omega)=\sqrt{\frac{\lambda_2}{2\pi\left(\frac{\lambda_1^2}{\sigma_2^2}+\frac{\lambda_2^2}{\sigma_1^2}\right)}},\ \Theta=-15\sum_{k=1}^{2}\mathbb{E}[X_k^{*3}]^2+9\sum_{k=1}^{2}\mathbb{E}[X_k^{*4}]-54, \tag{14}$$

$$\Xi_{0,1}=45\mathbb{E}[X_1^{*3}]^2-18\mathbb{E}[X_1^{*4}]+54,\ \Xi_{1,1}=18\mathbb{E}[X_1^{*3}]\mathbb{E}[X_2^{*3}],\ \Xi_{2,1}=45\mathbb{E}[X_2^{*3}]^2-18\mathbb{E}[X_2^{*4}]+54, \tag{15}$$

$$\Xi_{0,2}=-15\mathbb{E}[X_1^{*3}]^2+3\mathbb{E}[X_1^{*4}]-9, \Xi_{1,2}=\Xi_{3,2}=-6\mathbb{E}[X_1^{*3}]\mathbb{E}[X_2^{*3}], \Xi_{2,2}=0, \Xi_{4,2}=-15\mathbb{E}[X_2^{*3}]^2+3\mathbb{E}[X_2^{*4}]-9, \tag{16}$$

$$\Pi_k=\binom{2}{k}\ \mathbb{E}[X_1^{*3}]^k\mathbb{E}[X_2^{*3}]^{2-k},\ \text{for}\ \ k=0,1,2, \tag{17}$$

$$\begin{aligned}c_0=\frac{1}{36}&\left[\Theta+\frac{\lambda_1^2}{\lambda'\Sigma^{-1}\lambda}\sum_{k=0}^{2}\frac{\Xi_{k,1}}{\sigma_1^{2-k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^k+\frac{3\lambda_1^4}{(\lambda'\Sigma^{-1}\lambda)^2}\sum_{k=0}^{4}\frac{\Xi_{k,2}}{\sigma_1^{4-k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^k\right.\\&\left.+\frac{15\lambda_1^6}{(\lambda'\Sigma^{-1}\lambda)^3}\sum_{k=0}^{4}\frac{\Pi_k}{\sigma_1^{3k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^{3(2-k)}\right]+\frac{\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[X_k^{*3}]}{\sigma_k^3}}{2(\lambda'\Sigma^{-1}\lambda)^2}-\frac{1+2\sum_{k=1}^{2}\frac{\lambda_k\mathbb{E}[X_k^{*3}]}{\sigma_k}}{4\lambda'\Sigma^{-1}\lambda},\end{aligned} \tag{18}$$

$$c_1=\frac{\lambda_2\mathbb{E}[X_2^{*3}]}{\sigma_2}-1+\frac{\lambda_2^2-\frac{\mathbb{E}[X_1^{*3}]\lambda_1\lambda_2^2}{\sigma_1}-2\frac{\mathbb{E}[X_2^{*3}]\lambda_2^3}{\sigma_2}}{\sigma_2^2\lambda\Sigma^{-1}\lambda}+\frac{\lambda_2^2\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[X_k^{*3}]}{\sigma_k^3}}{\sigma_2^2(\lambda'\Sigma^{-1}\lambda)^2} \tag{19}$$

$$c_2=\frac{\lambda_2^2}{\sigma_2^2}\left(\frac{\lambda_2^2}{\sigma_2^2\lambda'\Sigma^{-1}\lambda}-1\right), \tag{20}$$

$$\gamma:=\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{E}\left[e^{-2(X_i+Y_j)}\ \left(\frac{\theta_2}{\theta_1}X_i-Y_j\right)\right]\ \text{and}\ \rho=\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{E}\left[e^{-2(X_i+Y_j)}\ \left(\frac{\theta_2}{\theta_1}X_i-Y_j\right)^2\right]. \tag{21}$$

For $i,\ j=1,\ 2$ and ν given in (7),

$$L_1(X_i,Y_j):=\frac{K(\nu)}{2\theta_2^{\frac{3}{2}}}\left[\left(1+c_1-2\frac{c_2}{\theta_1}+2\frac{c_2}{\theta_2}\right)v_{ij}+\frac{c_2}{\theta_2}v_{ij}^2\right],\quad L_2(X_i,Y_j)=c_2\ \frac{K(\nu)\ v_{ij}}{\theta_2^{\frac{5}{2}}},\quad v_{ij}:=Y_j-\frac{\theta_2}{\theta_1}X_i, \tag{22}$$

$$\tilde{L}_1=\frac{K(\nu)}{2\theta_1\sqrt{\theta_2}}\sum_{i=1}^{2}X_i\ e^{-X_i}+\sum_{i=1}^{2}\sum_{j=1}^{2}e^{-(X_i+Y_j)}\ L_1(X_i,Y_j),\qquad \tilde{L}_2=\sum_{i=1}^{2}\sum_{j=1}^{2}e^{-(X_i+Y_j)}\ L_2(X_i,Y_j), \tag{23}$$

$$A_1=\mathbb{E}\left[\tilde{L}_1^2\right],\ A_2=-\mathbb{E}\left[\tilde{L}_1\ \tilde{L}_2\right]\ \text{and}\ A_3=\mathbb{E}\left[\tilde{L}_2^2\right]. \tag{24}$$

When c_0, c_1, and c_2 are given by (18)–(20) associated to the distribution ν given by (7), we also define the two useful functions: For all $x\in\mathbb{R}$,

$$\eta(x)=\frac{K(\nu)}{2\sqrt{\theta_2}}\left[c_0-\frac{c_1}{\theta_1}+\frac{c_2}{\theta_2^2}+2\frac{c_2}{\theta_1^2}+\frac{(1-x)}{\theta_2}(1+c_1-2\frac{c_2}{\theta_1})+\frac{c_2(1-x)^2}{\theta_2^2}\right], \tag{25}$$

$$\tau(x)=\frac{A_1+2A_2x+A_3x^2}{1-\Phi(2,2)}+\frac{2(A_2+xA_3)\gamma+A_3\rho}{[1-\Phi(2,2)]^2}+\frac{A_3\gamma^2}{[1-\Phi(2,2)]^3}. \tag{26}$$

The constants are given above and Φ is given in (11).

3 | MAIN RESULTS

Once the expansion of the bi-renewal density function is established in Theorem 4.7 below, we can provide the asymptotic of the mean m and the variance $\mathbb{V}ar[N(x, y)]$ of the size our fragmentation tree and stat our main results.

Theorem 3.1 (Asymptotic of the mean of $N(x, y)$). *Let a be a positive number and η be given in (25). Under assumptions **(A)**, **(B)**, and **(C)**, we have*

$$\mathbb{M}(x^{\theta_1}, ax^{\theta_2}) = \frac{K(\nu)ax^{\theta_1+\theta_2}}{\sqrt{\theta_2 \ln x}} + \frac{a\eta(\ln a)x^{\theta_1+\theta_2}}{\ln^{\frac{3}{2}} x} + o\left(\frac{x^{\theta_1+\theta_2}}{\ln^{\frac{3}{2}} x}\right), \text{ as } x \to \infty. \tag{27}$$

Theorem 3.2 (Asymptotic of the variance of $N(x, y)$). *Let a be a positive number and τ be given in (26). Under assumptions **(A)**, **(B)**, and **(C)**, we have*

$$\mathbb{V}(x^{\theta_1}, ax^{\theta_2}) = \frac{\tau(\ln a)a^2x^{2(\theta_1+\theta_2)}}{\ln^3 x} + o\left(\frac{x^{2(\theta_1+\theta_2)}}{\ln^3 x}\right), \text{ as } x \to \infty. \tag{28}$$

4 | BI-RENEWAL THEORY

Bi-renewal theory is a fundamental method to study the behavior of $N(x, y)$. As a matter of fact, the one-dimensional process was studied by Feller,[9] Blackwell,[10,11] and Asmussen.[12] Model has been studied by numerous authors, namely, Hunter,[13,14] Mallora et al.,[15] Omey and Willekens,[16] and Mode.[17] Smith[8] developed the renewal theory, in particular the renewal density, for one dimension. Afterwards, these results have been extended by Mode,[17] who studied the case of a bi-dimensional renewal process. Unfortunately, the previous results are not sufficient to compute the second moment of a non-homogeneous fragmentation process. In this section, we prove a renewal theorem which is efficient for computing the variance. Under different assumptions, Hunter,[14] Mode,[17] Mallora et al[15] described the asymptotic behavior of the renewal density:

$$H(t_1, t_2) := \sum_{n=0}^{\infty} F^{*n}(t_1, t_2), \text{ with } F(t_1, t_2) = \nu([0, t_1] \times [0, t_2]) \text{ and } \nu \text{ is some probability measure.}$$

Their principal results are as follows: If (θ_1, θ_2) is the mean of the measure ν, then

- Mallora et al.[15]: for all $x,\ y \in \mathbb{R}$,

$$\lim_{t\to+\infty} \frac{H(tx,\ ty)}{t} = \min\left(\frac{x}{\theta_1}, \frac{y}{\theta_2}\right),$$

- Hunter[14]: Let $\varrho \in [0, 1[$ and I_0 be the modified Bessel function of the first kind of order zero. If ν has the probability density function,

$$u(t_1, t_2) = \frac{1}{\theta_1\theta_2(1-\varrho)} \exp\left(-\frac{\theta_1^{-1}t_1 + \theta_2^{-1}t_2}{1-\varrho}\right) I_0\left(\frac{2(\varrho\theta_1^{-1}\theta_2^{-1}t_1t_2)^{1/2}}{1-\varrho}\right),$$

$$H(\theta_1 t,\ \theta_2 t) = t - \sqrt{\frac{t(1-\varrho)}{\pi}} + o(\sqrt{t}), \text{ as } t \to \infty.$$

In Hunter,[14] it is also proved that

$$\frac{H(t_1,\ t_2)}{\sqrt{t_1t_2}} \to \min\left(\frac{\sqrt{K}}{\theta_1}, \frac{1}{\theta_2\sqrt{K}}\right), \text{ as } t_1,\ t_2 \to \infty, \text{ with } \frac{t_2}{t_1} \to K > 0.$$

Observe that in both previous situations, the approximation of H is given along the line

$$\{(tx, ty),\ t \in \mathbb{R}\},\ x,\ y \in \mathbb{R},$$

but their approximations do not give any information about the asymptotic behavior of the renewal density. Thus, the previous results are insufficient to give the behavior of $N(x, y)$ for our model.

4.1 | Local limit theorem for densities

This section exhibits new results that we managed to establish in the two dimensional case. Note that the proofs are based on technics different from those used in the one-dimensional case.

The proof of next Lemma is classical in dimension 1. The techniques and results used in dimension one remain true in higher dimension; see for instance Asmussen.[12, Theorem 2.4 pp. 146]

Lemma 4.1 (Existence and uniqueness). *Let ω and g be a finite measure and locally bounded function on $(0, \infty)^2$. Then, the function*

$$h(t_1, t_2) := \sum_{n=0}^{\infty} d\omega^{*n}(t_1, t_2)$$

is well defined, and $h * g$ is locally bounded and is the unique solution of the bi-dimensional renewal equation

$$f(t_1, t_2) = g(t_1, t_2) + (f * \omega)(t_1, t_2), \ (t_1, t_2) \in (0, \infty)^2. \tag{29}$$

Remark 4.2. By Lemma 4.1, the solution of the proper (when ω is a probability measure) renewal, Equation (12) is given by

$$M_e(t_1, \ t_2) = \sum_{n=0}^{\infty} (f * \nu^{*n})(t_1, \ t_2) = \sum_{n=0}^{\infty} \int_0^{t_1} \int_0^{t_2} e^{-(t_1 - s_1 + t_2 - s_2)} d\nu^{*n}(s_1, s_2),$$

where $f(s, z) = e^{-(s+z)}$ and ν is the probability measure given by Equation (7).

4.2 | Expansion of the bi-renewal density function

This lemma is a variant of the central limit theorem; it will be the key result for Corollary 4.4 which will be used to prove Theorem 4.7.

Lemma 4.3. *Let $X_1, \dots, X_n, \dots$ be a sequence of i.i.d random variables with common characteristic function φ satisfying condition (13). Assume that*

$$\mathbb{E}[X_i] = 0, \ \mathbb{V}ar[X_i] = 1 \ and \ \mathbb{E}[X_i^5] < \infty.$$

Then, the following holds.

1. *The characteristic function ψ_n of $Z_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$ satisfies*

$$\psi_n(u) = e^{-\frac{u^2}{2}} \left[1 + \frac{\Delta_1(u)}{\sqrt{n}} + \frac{\Delta_2(u)}{n} + \rho(u, n) \right], \ |\rho(u, n)| \leq D \frac{u^5}{n^{3/2}}, \ \frac{|u|}{\sqrt{n}} \leq C, \tag{30}$$

for some positive constant $C, \ D$ and

$$\Delta_1(u) = -\frac{i\mathbb{E}[X^3]}{6} u^3, \ \Delta_2(u) = \frac{u^4}{72} \left(3(\mathbb{E}[X^4] - 3) - \mathbb{E}^2[X^3] u^2 \right). \tag{31}$$

2. *For large n, the distribution Z_n is absolutely continuous, and its p.d.f d_n satisfies*

$$n \left[d_n(t) - \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} + \frac{\mathbb{E}[X^3]}{6\sqrt{2\pi} n^{1/2}} t \left(3 - t^2\right) e^{-\frac{t^2}{2}} \right] = d(t) + C(n, t),$$

where

$$d(t) := \frac{1}{144\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left(6(\mathbb{E}[X^4]-3)\left(3-6t^2+t^4\right)+2\mathbb{E}^2[X^3]\left(-15+45t^2-15t^4+t^6\right)\right) \text{ and } \lim_{n\to\infty}\sup_{t\in\mathbb{R}}|C(n,t)|=0. \tag{32}$$

Next corollary is a direct application of Lemma 4.3. It provides a local limit theorem for renewal densities, which is used in the proof of Theorem 4.7, if we denote by

$$\Lambda_1(t_1,t_2) = \frac{1}{6}\sum_{k=1}^{2}\mathbb{E}(X_k^3)(t_k^3-3t_k) \text{ and } \Lambda_2(t_1,t_2) = \frac{1}{72}\left[\Theta + \sum_{k=0}^{2}\Xi_{k,1}t_1^{2-k}t_2^k + \sum_{k=0}^{4}\Xi_{k,2}t_1^{4-k}t_2^k + \sum_{k=0}^{2}\Pi_k t_1^{3k}t_2^{3(2-k)}\right].$$

Corollary 4.4. *Consider a sequence of i.i.d continuous random vectors* $\left(X_{1,n}, X_{2,n}\right)_{n\in\mathbb{N}}$ *such that the random variables* $X_{1,n}$ *and* $X_{2,n}$ *are independent and such that their distribution satisfy the conditions of Lemma 4.3. Then, the probability density function* d_n *of the random vector* $(Z_{1,n}, Z_{2,n}) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_{1,j}, X_{2,j})$ *satisfies*

$$d_n(t_1,t_2) = \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\left[1+\frac{\Lambda_1(t_1,t_2)}{\sqrt{n}}+\frac{\Lambda_2(t_1,t_2)}{n}\right]+\Upsilon_n(t_1,t_2), \text{ uniformly in} (t_1,t_2), \text{as } n\to\infty, \tag{33}$$

where the function Υ_n *is given by*

$$\begin{aligned}\Upsilon_n(t_1,t_2) =& \frac{1}{2\pi n^{\frac{3}{2}}}\left[\frac{1}{\sqrt{2\pi}}\sum_{i,j=1,2,\ i\neq j} e^{-\frac{t_i^2}{2}}C_i(n,t_j) + \frac{e^{-\frac{t_1^2+t_2^2}{2}}}{432}\sum_{i=1,2} m_{3,i}(t_i^3-3t_i)\sum_{j=0}^{3}\beta_{i,j}t_i^{2j}\right] \\ &+ \frac{1}{12\pi n^2}\left[\frac{1}{\sqrt{2\pi}}\sum_{i,j=1,2,\ i\neq j} m_{3,i}(t_i^3-3t_i)e^{-\frac{t_i^2}{2}}C_i(n,t_j) + \frac{e^{-\frac{t_1^2+t_2^2}{2}}}{864}\sum_{j=0}^{3}\sum_{k=0}^{3}\beta_{1,j}\beta_{2,k}t_1^{2j}t_2^{2k}\right] \\ &+ \frac{1}{72(2\pi)^{\frac{3}{2}}n^{\frac{5}{2}}}\sum_{i,j=1,2,\ i\neq j} e^{-\frac{t_i^2}{2}}C_i(n,t_j)\sum_{j=0}^{3}\beta_{i,j}t_i^{2j} + \frac{C_1(n,t_1)C_2(n,t_2)}{4\pi^2 n^3},\end{aligned} \tag{34}$$

where for $i = 1, 2$, C_i *is given in (43) and* $m_{3,i}$ *(respectively* $m_{4,i}$*) denotes the value of the third (fourth) moment of* X_i*, and the value of* $\beta_{i,j}$ *is given by*

$$\beta_{i,0} = -15m_{3,i}^2+9m_{4,i}-27,\ \beta_{i,1} = 45m_{3,i}^2-18m_{4,i}+54\ \beta_{i,2} = -15m_{3,i}^2+3m_{4,i}-9, \text{ and } \beta_{i,3} = m_{3,i}^2.$$

Since the p.d.f. of $(Z_{1,n}, Z_{2,n})$ is of the form $d_{1,n}(t_1)d_{2,n}(t_2)$, there is no need to prove last corollary which only require some tedious but straightforward computations.

Remark 4.5. If $X_{1,n}$ and $X_{2,n}$ are not independent, then a bi-variate analog of Condition 13 on the characteristic function

$$\Psi(v_1,v_2) = \mathbb{E}[e^{i(v_1X_{1,1}+v_2X_{2,1})}],\ (v_1,v_2)\in\mathbb{R}^2$$

could be sufficient to have the same conclusion than Corollary 4.4. For instance, one may assume

a. $|\Psi(v_1,v_2)| \le \frac{c}{|v_1|^{\alpha_1}}$, if $|v_2| < M \le |v_1|$,
b. $|\Psi(v_1,v_2)| \le \frac{c}{|v_2|^{\alpha_2}}$, if $|v_1| < M \le |v_2|$,
c. $|\Psi(v_1,v_2)| \le \frac{c}{|v_1|^{\alpha_1}|v_2|^{\alpha_2}}$, if $|v_1|,\ |v_2| \ge M$,

for some non-negative numbers $\alpha_1,\ \alpha_2,\ c$.

From Corollary 4.4, we conclude that as n tends to infinity, we have uniformly on (t_1, t_2),

$$d_n(t) \to \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}.$$

The next proposition will be devoted to prove that if we multiply the two sides by $|t_k|^r$ with $r \in [0, 5]$, the convergence still uniform on (t_1, t_2). This result was stated by Mode[17] for all $r \in [0, 2]$; we generalize it for all $r \in [0, 5]$ if the random vectors have finite moment of order 5.

Proposition 4.6. *Under assumptions and the notations of Corollary 4.4, we have for all* $0 \le m \le 5$, *uniformly on* $(t_1, t_2) \in \mathbb{R}^2$:

$$\lim_{n\to+\infty} |t_k|^m d_n(t_1, t_2) = |t_k|^m \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}.$$

In the following theorem, we use Corollary 4.4 to obtain a control of the bi-renewal density function.

Theorem 4.7. *Let* $(\boldsymbol{X}_n)_n = \left(X_{1,n}, X_{2,n}\right)_n$ *be a sequence of i.i.d and absolutely continuous random vectors in* $[0, \infty)^2$ *satisfying the assumptions of Corollary 4.4. Let*

$$\lambda := (\lambda_1, \lambda_2) = \mathbb{E}[\boldsymbol{X}_1],\ \Sigma := \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} = \mathbb{C}ov[\boldsymbol{X}_1] \text{ and } \boldsymbol{S}_n := \sum_{k=1}^{n} \boldsymbol{X}_k,$$

h_n be the probability density function of $\boldsymbol{S}_n$,

$$h(t_1, t_2) := \sum_{n=1}^{\infty} h_n(t_1, t_2),$$

and

$$C(t_1, t_2) = \frac{K(\mu_X)\lambda_1}{2}\left[c_0 + c_1\, F(t_1, t_2) + c_2\, F(t_1, t_2)^2\right],\ F(t_1, t_2) := \frac{t_1}{\lambda_1} - \frac{t_2}{\lambda_2}, \tag{35}$$

where μ_X is the distribution of X, $K(\mu_X)$ is given by (14), and c_i, $i = 0, 1, 2$ are given by (18)–(20), if

$$\lim_{t_k\to+\infty} t_k^{\frac{3}{2}}\, h_n(t_1, t_2) = 0, \text{ for all } n \ge 1 \text{ and } k = 1, 2. \tag{36}$$

For every fixed $B > 0$ fixed, we have

$$\sqrt{t_2}\, h(t_1, t_2) = K(\mu_X) + \frac{C(t_1, t_2)}{t_1} + o\left(\frac{1}{t_1}\right),\ \text{as } t_1, t_2 \to \infty,\ |F(t_1, t_2)| \le B.$$

An immediate consequence of last theorem is the following result.

Corollary 4.8. *Under the assumptions of Corollary 4.4 with $\lambda_1 = \lambda_2 := \lambda$, the following assertions hold:*

a) We have

$$\sqrt{t}\, h(t, t) = K(\omega) + \frac{K(\omega)\ \lambda\ c_0}{2t} + o\left(\frac{1}{t}\right),\ t \to \infty.$$

b) With C given by (35) and for every fixed $B > 0$ fixed, we have

$$\sqrt{\lambda t_2}\, h\left(\lambda t_1, \lambda t_2\right) = K(\omega) + \frac{K(\omega)\ C(\lambda t_1, \lambda t_2)}{\lambda t_1} + o\left(\frac{1}{t_1}\right),\ \text{as } t_1, t_2 \to \infty,\ |t_1 - t_2| \le B.$$

Example 4.9. Let (X_1, X_2) be a r.v. with bi-variate exponential distribution with parameters $r_1,\ r_2 > 0$, belongs to the set, i.e., with p.d.f.

$$r_1\ r_2\ e^{-(r_1 t_1 + r_2 t_2)},\ (t_1, t_2) \in (0, \infty)^2.$$

The characteristic function satisfies $\Psi(t_1, t_2) = \Psi_1(t_1)\Psi_2(t_2)$, and

$$\Psi_k(t_k) = \frac{r_k}{(it_k - r_k)},\ |\Psi_k(t_k)| = \frac{r_k}{\sqrt{t_k^2 + r_k^2}} \leq \frac{r_k}{t_k} \Rightarrow \Psi_k \in \mathbb{J}_1.$$

The mean of (X_1, X_2) is given by $(\frac{1}{r_1}, \frac{1}{r_2})$, and the covariance matrix is a definite positive matrix given by $\Sigma = \begin{pmatrix} r_1^{-2} & 0 \\ 0 & r_2^{-2} \end{pmatrix}$. The nth convolution, corresponding to the renewal density, is given by

$$h(t_1, t_2) := \sum_{n=1}^{+\infty} \frac{(r_1 r_2)^n (t_1 t_2)^{n-1}}{(n-1)!^2}\ e^{-(r_1 t_1 + r_2 t_2)} = r_1\ r_2\ e^{-(r_1 t_1 + r_2 t_2)}\ I_0(2\sqrt{r_1 r_2 t_1 t_2}),$$

where I_0 is the modified Bessel function of the first kind, cf. Abramowitz and Stegun,[18] which satisfies

$$I_0(t) = \frac{e^t}{\sqrt{2\pi t}} \left[1 + \frac{1}{8t} + o\left(\frac{1}{t}\right)\right],\ \text{as}\ t \to \infty. \tag{37}$$

If $t_1,\ t_2 \to \infty$ and $|r_1 t_1 - r_2 t_2| \leq B$, for some positive number B, then

$$\sqrt{r_1 t_1} - \sqrt{r_2 t_2} \to 0,\ \frac{t_2}{t_1} \to \frac{r_1}{r_2}\ \text{and}\ \sqrt{t_2} h(t_1, t_2) = \frac{r_2^{1/2} r_1}{\sqrt{2\pi}} \left[1 + \frac{1}{8 r_1 t_1} + o\left(\frac{1}{t_1}\right)\right].$$

5 | THE PROOFS

The following Lemma is an immediate consequence of Mode[17, Lemma 3.1] and will be used in the sequel.

Lemma 5.1. *Let $\xi,\ G : (0, \infty)^2 \to \mathbb{R}$ such that ξ is integrable and G is a uniformly bounded function,*

$$\lim_{t_1,\ t_2 \to \infty,\ |t_1 - t_2| \leq B} G(t_1, t_2) \to L,\ \textit{for every}\ B > 0.$$

Then, for any real number α,

$$\lim_{t \to +\infty} \int_0^t \int_0^{t+\alpha} \xi(s_1, s_2)\ G(t - s_1, t + \alpha - s_2)\ ds_2\ ds_1 = L \int_0^\infty \int_0^\infty \xi(s_1, s_2) ds_1 ds_2.$$

Proof. Theorem 3.1. The function $\mathbb{M}_e(t_1, t_2) := e^{-(t_1 + t_2)}\ \mathbb{M}(e^{t_1}, e^{t_2})$ satisfies (12) and is given by

$$\mathbb{M}_e(t_1, t_2) = (e^{-(t_1 + t_2)} * h)(t_1, t_2),\ \text{where}\ \ h(t_1, t_2) = \sum_{n=0}^{\infty} h_n(t_1, t_2)\ \text{with}\ h_n\ \text{given by}\ (9).$$

Let us denote $\tilde{h}(t_1, t_2) = h(\theta_1 t_1, \theta_2 t_2)$ and $\tilde{C}(t_1, t_2) = C(\theta_1 t_1, \theta_2 t_2)$ where C is given by (35). Then, for all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{M}_e(\theta_1 t, \theta_2 t + \alpha) &= \int_0^{\theta_1 t} \int_0^{\theta_2 t+\alpha} e^{-(s_1+s_2)} h(\theta_1 t - s_1, \theta_2 t + \alpha - s_2) ds_1 ds_2 \\ &= \theta_1\theta_2 \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} e^{-(\theta_1 u+\theta_2 v)} \tilde{h}(t-u, t+\frac{\alpha}{\theta_2} - v) du dv.\end{aligned}$$

Then,

$$\hat{M}(t) := t\sqrt{(t+\frac{\alpha}{\theta_2})}\mathbb{M}_e(\theta_1 t, \theta_2 t + \alpha) \tag{38}$$

can be written as

$$\begin{aligned}\hat{M}(t) = I_1(t) + I_2(t) + \theta_1 K(v) t\sqrt{\theta_2 t+\alpha} \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{-(\theta_1 u+\theta_2 v)}}{\sqrt{t+\frac{\alpha}{\theta_2}-v}} du dv \\ + \sqrt{\theta_2 t+\alpha} \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{-(\theta_1 u+\theta_2 v)} \tilde{C}(t-u, t+\frac{\alpha}{\theta_2}-v)}{\sqrt{t+\frac{\alpha}{\theta_2}-v}} du dv,\end{aligned}$$

where

$$I_1(t) = \theta_1\theta_2 \sqrt{t+\frac{\alpha}{\theta_2}} \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} \left[(t-u)\tilde{h}(t-u, t+\frac{\alpha}{\theta_2}-v) - \frac{K(v)(t-u)}{\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}} - \frac{\tilde{C}(t-u, t+\frac{\alpha}{\theta_2}-v)}{\theta_1\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}} \right] e^{-(\theta_1 u+\theta_2 v)} du dv$$

$$I_2(t) = \theta_1\theta_2 \sqrt{t+\frac{\alpha}{\theta_2}} \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} u\, e^{-(\theta_1 u+\theta_2 v)} \left[\tilde{h}(t-u, t+\frac{\alpha}{\theta_2}-v) - \frac{K(v)}{\sqrt{\theta_2(t+\frac{\alpha}{\theta_2}-v)}} \right] du dv.$$

We have $I_1(t) = o(1)$, as $t \to +\infty$. Indeed, using the fact that

$$v \in [0, t+\frac{\alpha}{\theta_2}] \Rightarrow \sqrt{t+\frac{\alpha}{\theta_2}} \le \sqrt{t+\frac{\alpha}{\theta_2}-v} + \sqrt{v},$$

we obtain, with

$$G_1(t_1, t_2) := \sqrt{t_2}\, t_1\, \tilde{h}(t_1, t_2) - \frac{K(v)t_1}{\sqrt{\theta_2}} - \frac{\tilde{C}(t_1, t_2)}{\theta_1\sqrt{\theta_2}},\ G_2(t_1, t_2) := t_1\, \tilde{h}(t_1, t_2) - \frac{K(v)t_1}{\sqrt{\theta_2 t_2}} - \frac{\tilde{C}(t_1, t_2)}{\theta_1\sqrt{\theta_2 t_2}},$$

that

$$|I_1(t)| \le \theta_1\theta_2 \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} \left[\left| G_1\left(t-u, t+\frac{\alpha}{\theta_2}-v\right) \right| + \sqrt{v} \left| G_2\left(t-u, t+\frac{\alpha}{\theta_2}-v\right) \right| \right] e^{-(\theta_1 u+\theta_2 v)} du dv.$$

Assumptions (**A**), (**B**), and (**C**) guarantee the hypothesis of Theorem 4.7, then as $t_1,\ t_2 \to \infty$ and $|t_1 - t_2| \le B$, we have $G_1(t_1, t_2) \to 0$ and $G_2(t_1, t_2) \to 0$. Furthermore, ξ_1 and ξ_2 are integrable on $(0, \infty)^2$; we conclude by Lemma 5.1 that as t tends to infinity, $I_1(t) \to 0$. By similar argument, we prove that as $t \to +\infty,\ I_2(t) = o(1)$. Thus,

$$\hat{M}(t) = \sqrt{\theta_2 t+\alpha} e^{-(\theta_1+\theta_2)t-\alpha} \int_0^{t} \int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{\theta_1 s_1+\theta_2 s_2} \tilde{C}(s_1, s_2)}{\sqrt{s_2}} ds_1 ds_2 + 2tK(v)\sqrt{t+\frac{\alpha}{\theta_2}}(1-e^{-\theta_1 t}) Daw(\sqrt{\theta_2 t+\alpha}) + o(1),$$

where Daw is the Dawson's integral[18, p. 295 and p.319] given by

$$Daw(t) = e^{-t^2}\int_0^t e^{u^2}du.$$

Moreover, as $t \to +\infty$, we have

$$Daw(\sqrt{\theta_2 t+\alpha}) = \frac{1}{2\sqrt{\theta_2 t+\alpha}} + \frac{1}{4(\theta_2 t+\alpha)^{\frac{3}{2}}} + \frac{3}{8(\theta_2 t+\alpha)^{\frac{5}{2}}} + o\left(\frac{1}{t^{\frac{5}{2}}}\right) = \frac{1}{2\sqrt{\theta_2 t}} + \frac{1-\alpha}{4(\theta_2 t)^{\frac{3}{2}}} + \frac{3(\alpha^2-2\alpha+2)}{16(\theta_2 t)^{\frac{5}{2}}} + o\left(\frac{1}{t^{\frac{5}{2}}}\right).$$

We deduce that

$$2t\sqrt{t+\frac{\alpha}{\theta_2}}(1-e^{-\theta_1 t})Daw(\sqrt{\theta_2 t+\alpha}) = \frac{t}{\sqrt{\theta_2}} + \frac{1}{2}\theta_2^{\frac{3}{2}} + o(1),$$

which leads to

$$\hat{M}(t) = \sqrt{\theta_2 t+\alpha}\, e^{-(\theta_1+\theta_2)t-\alpha}\int_0^t\int_0^{t+\frac{\alpha}{\theta_2}} \frac{e^{\theta_1 s_1+\theta_2 s_2}\tilde{C}(s_1,s_2)}{\sqrt{s_2}}ds_1 ds_2 + \frac{tK(\nu)}{\sqrt{\theta_2}} + \frac{K(\nu)}{2\theta_2^{\frac{3}{2}}} + o(1).$$

Explicitly $\tilde{C}$ and integrating, we get

$$\hat{M}(t) = \frac{K(\nu)}{2\sqrt{\theta_2 t}}\sqrt{t+\frac{\alpha}{\theta_2}}\left[c_0 - \frac{c_1}{\theta_1} + \frac{(1-\alpha)}{\theta_2}(c_1 - 2\frac{c_2}{\theta_1}) + \frac{2c_2}{\theta_1^2} + \frac{c_2}{\theta_2^2} + \frac{c_2(\alpha-1)^2}{\theta_2^2}\right] + \frac{tK(\nu)}{\sqrt{\theta_2}} + \frac{K(\nu)}{2\theta_2^{\frac{3}{2}}} + o(1).$$

By (51) and (5), we conclude that

$$\mathbb{M}_e(\theta_1 t, \theta_2 t+\alpha) = \frac{K(\nu)}{\sqrt{\theta_2 t}} + \frac{\eta(\alpha)}{t^{\frac{3}{2}}} + o\left(\frac{1}{t^{\frac{3}{2}}}\right), \text{ as } t\to+\infty, \text{ where } \eta \text{ is given by (25).}$$

□

Proof. Theorem 3.2. Let us define the function V by

$$\mathbb{V}(u,v) := \mathbb{V}ar[N(u,v)] \text{ and } \mathbb{V}_1(t_1,t_2) := \mathbb{V}(e^{t_1}, e^{t_2}).$$

By conditioning, we have

$$\begin{aligned}\mathbb{E}\left[(N(x,y)-\mathbb{M}(x,y))^2|\sigma(X_1,Y_1)\right] &= \mathbb{E}\left[\left(1+\sum_{i=1}^{2}\sum_{j=1}^{2}N(x\,e^{-X_i}, y\,e^{-Y_j}) - \mathbb{M}(x,y)\right)^2|\sigma(X_1,Y_1)\right]\\ &= \sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{V}(x\,e^{-X_i}, y\,e^{-Y_j}) + \left(1-\mathbb{M}(x,y)+\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{M}(x\,e^{-X_i}, y\,e^{-Y_j})\right)^2.\end{aligned}$$

Taking the expectations in both sides of last equality, we see that $\mathbb{V}_1$ satisfies the bi-variate renewal equation

$$\mathbb{V}_1(t_1,t_2) = (\mathbb{V}_1(s_1,s_2)*\mu)(t_1,t_2) + k(t_1,t_2), \text{ where } k(t_1,t_2) = \mathbb{E}\left[\left(1-\mathbb{M}(e^{t_1},e^{t_2}) + \sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{M}(e^{t_1-X_i}, e^{t_2-Y_j})\right)^2\right]. \quad (39)$$

The function $\mathbb{V}_e(t_1,t_2) := e^{-(t_1+t_2)}\mathbb{V}_1(t_1,t_2)$ also satisfies a renewal equation:

$$\mathbb{V}_e(t_1,t_2) = (\mathbb{V}_e*\nu)(t_1,t_2) + e^{-(t_1+t_2)}k(t_1,t_2).$$

By Lemma 4.1, it follows that

$$\mathbb{V}_e(\theta_1 t, \theta_2 t+\alpha) = (h * e^{-(s_1+s_2)}k)(\theta_1\ t, \theta_2\ t+\alpha)$$
$$= e^{-[(\theta_1+\theta_2)t+\alpha]}\sum_{n=0}^{\infty}\mathbb{E}\left[e^{\left(S_n^{(1)}+S_n^{(2)}\right)}k\left(\theta_1 t - S_n^{(1)}\ ,\ \theta_2 t+\alpha - S_n^{(2)}\right)\mathbf{1}_{\{S_n^{(1)}\le\theta_1 t\ ,\ S_n^{(2)}\le\theta_2 t+\alpha\}}\right],$$

where the vector $\left(S_n^{(1)},\ S_n^{(2)}\right)$ has the distribution ν^{*n}. Using the expression (27) of $\mathbb{M}$ and putting there, $x = e^{t-\frac{X_i}{\theta_1}}$ and $a = e^{(\theta_2-\theta_1)t-Y_j+\alpha}$, we obtain that conditionally on X_i and Y_j, and on the event $\{X_i < \theta_1 t\}$,

$$\mathbb{M}\left(e^{\theta_1\ t-X_i}, e^{\theta_2 t-Y_j+\alpha}\right) = e^{(\theta_1+\theta_2)t+\alpha-X_i-Y_j}\left[\frac{K(\nu)}{\sqrt{\theta_2(t-\frac{X_i}{\theta_1})}} + \frac{\eta(\frac{\theta_2}{\theta_1}X_i - Y_j+\alpha)}{(t-\frac{X_i}{\theta_1})^{3/2}} + o\left(\frac{1}{(t-\frac{X_i}{\theta_1})^{3/2}}\right)\right],\ \text{as } t\to\infty. \tag{40}$$

At this step, one can use the inequality in (5) and deduce that

$$\mathbb{E}[\mathbb{M}\left(e^{\theta_1\ t-X_i}, e^{\theta_2 t-Y_j+\alpha}\right)] \le \mathbb{E}\left[\mathbb{M}(e^{\theta_1\ t-X_i})\ \mathbb{M}(e^{\theta_2 t-Y_j+\alpha})\right] \le \mathbb{E}\left[\mathbb{M}(e^{\theta_1\ t-X_i})\right]\ \mathbb{E}\left[\mathbb{M}(e^{\theta_2 t-Y_j+\alpha})\right]$$
$$\le \mathbb{M}(e^{\theta_1\ t})\ \mathbb{M}(e^{\theta_2 t+\alpha}) < \infty.$$

Thus, we can decompose $\mathbb{M}\left(e^{\theta_1\ t-X_i}, e^{\theta_2 t-Y_j+\alpha}\right)$ into the form

$$e^{-(\theta_1+\theta_2)t-\alpha+X_i+Y_j}\ \mathbb{M}\left(e^{\theta_1\ t-X_i}, e^{\theta_2 t-Y_j+\alpha}\right) = \frac{K(\nu)}{\sqrt{\theta_2 t}}\left(1+\frac{X_i}{2\theta_1 t}\right) + \frac{1}{t^{3/2}}\left(\eta(\alpha) + L_1(X_i, Y_j) - \alpha L_2(X_i, Y_j)\right) + R_{i,j}(t),$$

where $L_1(X_i, Y_j)$ and $L_2(X_i, Y_j)$ are given by (22). The remainder r.v. $R_{i,j}(t)$, satisfies

$$\mathbb{E}[e^{-X_i-Y_j}\ R_{i,j}(t)] = o\left(\frac{1}{t^{\frac{3}{2}}}\right), \tag{41}$$

Moreover, by summing (40), we deduce that

$$e^{-(\theta_1+\theta_2)t-\alpha}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbb{M}(e^{\theta_1 t-X_i}, e^{\theta_2 t+\alpha-Y_j}) - \mathbb{M}(e^{\theta_1 t}, e^{\theta_2 t+\alpha})\right) = \frac{\tilde{L}_1 - \alpha\tilde{L}_2}{t^{\frac{3}{2}}} + o\left(\frac{1}{t^{\frac{3}{2}}}\right),$$

where $\tilde{L}_1$ and $\tilde{L}_2$ are given by (23). Consequently, the variance of the last expression equals to

$$e^{-2(\theta_1+\theta_2)t-2\alpha}\ k(\theta_1 t, \theta_2 t+\alpha) = \frac{1}{t^3}\left(A_1 + 2A_2\alpha + A_3\alpha^2\right) + o\left(\frac{1}{t^3}\right),$$

where $A_1,\ A_2$, and A_3 are given in (24). Conditioning on $\left(S_n^{(1)}, S_n^{(2)}\right)$, and replacing t by $t - S_n^{(1)}/\theta_1$ and α by $S_n^{(1)}\theta_2/\theta_1 - S_n^{(2)} + \alpha$, we obtain

$$k(\theta_1 t - S_n^{(1)}, \theta_2 t+\alpha - S_n^{(2)}) = t^{-3}e^{2(\theta_1+\theta_2)t+2\alpha-2(S_n^{(1)}+S_n^{(2)})}\left[(A_1 + 2A_2\alpha + A_3\alpha^2)\right.$$
$$\left. +2(A_2+\alpha A_3)\left(\frac{\theta_2}{\theta_1}S_n^{(1)} - S_n^{(2)}\right) + A_3\left(\frac{\theta_2}{\theta_1}S_n^{(1)} - S_n^{(2)}\right)^2\right] + e^{-2(S_n^{(1)}+S_n^{(2)})}o\left(\frac{e^{2(\theta_1+\theta_2)t+2\alpha}}{t^3}\right).$$

Finally, we get

$$\mathbb{V}(\theta_1 t, \theta_2 t+\alpha) = t^{-3}\, e^{(\theta_1+\theta_2)t+\alpha}\tau(\alpha) + o\left(\frac{e^{(\theta_1+\theta_2)t+\alpha}}{t^3}\right),$$

where τ is given by (26). □

Proof. Lemma 4.3 Let $m_i := \mathbb{E}[X^i]$. To prove this lemma, we need some Taylor expansion for $\ln(1+u)$ and φ at the neighborhood of 0. On one hand, we have the following:

a) Fix $\varepsilon \in (0,1)$. For all $|x| \in (0, 1-\varepsilon)$, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^5 R_0(x),\ e^x = 1 + x + \frac{x^2}{2} + x^3\, R_1(x),\ |R_0(x)| \le 1/\varepsilon^5,\ |R_1(x)| \le e^{1-\varepsilon}.$$

On the other hand,

$$\varphi(\frac{u}{\sqrt{n}}) = 1 - \frac{u^2}{2n} - i\frac{m_3}{6n^{\frac{3}{2}}}u^3 + \frac{m_4}{24n^2}u^4 + \frac{u^5}{n^{\frac{5}{2}}}R_2(u/\sqrt{n}),\ |R_2(u/\sqrt{n})| \le m_5. \tag{42}$$

Thus, there exists $C_0 > 0$, such that for all $|u|/\sqrt{n} \le C_0\,(1-\varepsilon)$, we have $\left|\varphi\left(\frac{u}{\sqrt{n}}\right) - 1\right| \le (1-\varepsilon)^2 \le (1-\varepsilon)$ and then

$$n\ln\left(\varphi(\frac{u}{\sqrt{n}})\right) = -\frac{u^2}{2} - i\frac{m_3}{6n^{\frac{1}{2}}}u^3 + \frac{1}{24n}(m_4 - 3)\,u^4 + \frac{u^5}{n^{\frac{3}{2}}}R_3(u/\sqrt{n}),\ |R_3(u/\sqrt{n})| \le D_3.$$

Finally, we deduce that there exists C_2 such that for all $|u|/\sqrt{n} \le C_2(1-\varepsilon)$, we have

$$\psi_n(u) = e^{-\frac{u^2}{2}}\left(1 - i\frac{m_3}{6n^{\frac{1}{2}}}u^3 - \frac{u^4}{72n}\left[m_3^2u^2 - 3\,(m_4-3)\right] + \frac{u^5}{n^{\frac{3}{2}}}R_4(u/\sqrt{n})\right),\ |R_4(u/\sqrt{n})| \le D_4.$$

The latter shows (30). Let denote by $m_k = \mathbb{E}[X^k],\ k \ge 1$. The following equality is straightforward:

$$\begin{aligned}\psi_n(u) = {} & e^{-\frac{u^2}{2}}\left(1 - i\frac{m_3}{6n^{\frac{1}{2}}}u^3 - \frac{u^4}{72n}\left[m_3^2u^2 - 3\,(m_4-3)\right] + \frac{u^5}{n^{\frac{3}{2}}}R_4(u/\sqrt{n})\right)\mathbb{1}_{\frac{|u|}{\sqrt{n}} \le C_2(1-\varepsilon)} \\ & + \psi_n(u)\mathbb{1}_{\frac{|u|}{\sqrt{n}} \ge M} + \psi_n(u)\mathbb{1}_{C_2(1-\varepsilon) \le \frac{|u|}{\sqrt{n}} \le M}.\end{aligned}$$

Condition (13) and Fourier inversion theorem insure that for $n\alpha > 1$, Z_n has an absolutely continuous distribution with p.d.f. d_n expressed by

$$\begin{aligned}d_n(t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}}\psi_n(u)e^{-iut}du = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} - \frac{1}{2\pi}\int_{\frac{|u|}{\sqrt{n}} > C_2(1-\varepsilon)} e^{-\frac{u^2}{2}}e^{-iut}du - i\frac{m_3}{12\pi n^{1/2}}\int_{|\frac{u}{\sqrt{n}}| \le C_2(1-\varepsilon)} u^3 e^{-\frac{u^2}{2}}e^{-iut}du \\ & + \frac{1}{144\pi n}\int_{\frac{|u|}{\sqrt{n}}| \le C_2(1-\varepsilon)}\left[3(m_4-3) - m_3^2u^2\right]u^4 e^{-\frac{u^2}{2}}e^{-iut}du + \frac{1}{2\pi n^{3/2}}\int_{\frac{|u|}{\sqrt{n}} \le C_2(1-\varepsilon)} R_4(u)e^{-\frac{u^2}{2}}e^{-iut}du \\ & + \frac{1}{2\pi}\int_{\frac{|u|}{\sqrt{n}} \ge M}\psi_n(u)e^{-iut}du + \frac{1}{2\pi}\int_{C_2(1-\varepsilon) \le \frac{|u|}{\sqrt{n}} \le M}\psi_n(u)e^{-iut}du.\end{aligned}$$

Using condition (13) again, together with[19, Theorem 2.1.4 p. 18,] we obtain that there exists $\lambda \in (0,1)$ such that for all $v \in [1-\varepsilon, M]$ we have $|\varphi(v)| \le \lambda$. Then, for some constants E_i such that, we have

i) $\left|\int_{\frac{|u|}{\sqrt{n}} > C_2(1-\varepsilon)} e^{-\frac{u^2}{2}}e^{-iut}du\right| \le \frac{E_1}{n^2}$

ii) $\left|\frac{1}{n^{3/2}}\int_{\frac{|u|}{\sqrt{n}}| \le C_2(1-\varepsilon)} R_4(u)e^{-\frac{u^2}{2}}e^{-iut}du\right| \le \frac{E_2}{n^{\frac{3}{2}}}$.

iii) $\left|\int_{\frac{|u|}{\sqrt{n}} \ge M}\psi_n(u)e^{-iut}du\right| \le 2\int_{M\sqrt{n}}^{+\infty}\left(\frac{\sqrt{n}}{u}\right)^{n\alpha}du = \frac{E_3\sqrt{n}}{M^{n\alpha}}$

iv) $\left|\int_{C_2(1-\varepsilon) \le \frac{|u|}{\sqrt{n}} \le M}\psi_n(u)e^{-iut}du\right| \le 2\int_{(1-\varepsilon)\sqrt{n}}^{M\sqrt{n}}\lambda^n du = E_4\sqrt{n}\lambda^n$.

Since

$$\int_{\mathbb{R}} u^3 e^{-\frac{u^2}{2}} e^{-iut} du = \sqrt{2\pi} t \left(3 - t^2\right) e^{-\frac{t^2}{2}}$$

and

$$\int_{\mathbb{R}} \left[3(m_4 - 3) - m_3^2 u^2\right] u^4 e^{-\frac{u^2}{2}} e^{-iut} du = \sqrt{\frac{\pi}{2}} e^{-\frac{t^2}{2}} \left(6(m_4 - 3)\left(3 - 6t^2 + t^4\right) + 2m_3^2\left(-15 + 45t^2 - 15t^4 + t^6\right)\right),$$

then we conclude that

$$n\left[d_n(t) - \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} + i\frac{m_3}{12\pi n^{1/2}} \int_{\mathbb{R}} u^3 e^{-\frac{u^2}{2}} e^{-iut} du\right] = \frac{1}{144\pi} \int_{\mathbb{R}} \left[3(m_4 - 3) - m_3^2 u^2\right] u^4 e^{-\frac{u^2}{2}} e^{-iut} du + \frac{C(n, t)}{2\pi\sqrt{n}}, \tag{43}$$

and then,

$$n\left[d_n(t) - \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} + \frac{m_3}{6\sqrt{2\pi} n^{1/2}} t\left(3 - t^2\right) e^{-\frac{t^2}{2}}\right] = d(t) + \frac{C(n, t)}{2\pi\sqrt{n}},$$

where the bounded functions $C(n, .)$ satisfy

$$\sup_{t\in\mathbb{R}} |C(n, t)| \le \frac{E_1}{n} + \frac{E_2}{\sqrt{n}} + \frac{E_3\, n^{3/2}}{M^{n\alpha}} + E_4\, n^2 \lambda^n \to 0.$$

□

Proof. Proposition 4.6. We start to show that the conclusion of Proposition 4.6 is true for $m = 5$. For this end, let us denote $f_{n,k}(t_1, t_2) = t_k^5 d_n(t_1, t_2),\ k = 1, 2$. Since the moments of order 5 of our variables are finite, then obviously, $f_{n,k}$ is integrable, and the Fourier transform of $f_{n,k}$ is given by

$$\hat{f}_{n,k}(t_1, t_2) = -i\frac{\partial^5 \Psi_n}{\partial t_k^5}(-t_1, -t_2).$$

Recall that

$$\Psi_n(t_1, t_2) = \Psi_{n,1}(t_1)\Psi_{n,2}(t_2) = \varphi_1\left(\frac{t_1}{\sqrt{n}}\right)^n \varphi_2\left(\frac{t_2}{\sqrt{n}}\right)^n,$$

where for $k = 1, 2$, $\Psi_{n,k}$ (respectively φ_k) denotes the characteristic function of the random variable $n^{-1/2}\sum_{j=1}^{n} X_{k,j}$ (respectively of $X_{k,n}$). A simple computation the expression gives, for $1 \le k \ne l \le 2$,

$$\begin{aligned}
\frac{\partial^5 \Psi_n}{\partial t_k^5}(t_1, t_2) = \Psi_{n,l}(t_l)\Psi_{n,k}^{(5)}(t_k) = \varphi_l\left(\frac{t_l}{\sqrt{n}}\right)^n \Bigg[&\frac{1}{n^{\frac{3}{2}}} \varphi_k^{(5)}(\frac{t_k}{\sqrt{n}})\varphi_k^{n-1}(\frac{t_k}{\sqrt{n}}) + \frac{5(n-1)}{n^{\frac{3}{2}}} \varphi_k^{(4)}(\frac{t_k}{\sqrt{n}})\varphi_k'(\frac{t_k}{\sqrt{n}})\varphi_k^{n-2}(\frac{t_k}{\sqrt{n}}) \\
&+ \frac{10(n-1)}{n^{\frac{3}{2}}} \varphi_k^{(3)}(\frac{t_k}{\sqrt{n}})\varphi_k^{(2)}(\frac{t_k}{\sqrt{n}})\varphi_k^{n-2}(\frac{t_k}{\sqrt{n}}) + \frac{10(n-1)(n-2)}{n^{\frac{3}{2}}} \varphi_k^{(3)}(\frac{t_k}{\sqrt{n}}) \left[\varphi_k'(\frac{t_k}{\sqrt{n}})\right]^2 \varphi_k^{n-3}(\frac{t_k}{\sqrt{n}}) \\
&+ \frac{15(n-1)(n-2)}{n^{\frac{3}{2}}} \varphi_k'(\frac{t_k}{\sqrt{n}}) \left[\varphi_k^{(2)}(\frac{t_k}{\sqrt{n}})\right]^2 \varphi_k^{n-3}(\frac{t_k}{\sqrt{n}}) \\
&+ \frac{10(n-1)(n-2)(n-3)}{n^{\frac{3}{2}}} \varphi_k^{(2)}(\frac{t_k}{\sqrt{n}}) \left[\varphi_k'(\frac{t_k}{\sqrt{n}})\right]^3 \varphi_k^{n-4}(\frac{t_k}{\sqrt{n}}) \\
&+ \frac{(n-1)(n-2)(n-3)(n-4)}{n^{\frac{3}{2}}} \left[\varphi_k'(\frac{t_k}{\sqrt{n}})\right]^5 \varphi_k^{n-5}(\frac{t_k}{\sqrt{n}})\Bigg].
\end{aligned} \tag{44}$$

Observing that the function and that the function $x \mapsto x^{x^2}$ is non-increasing on $(0, 1/\sqrt{e})$, then choosing the constant M in (13) bigger than $\sqrt{e}\max(c_1^{1/\alpha_1}, c_2^{1/\alpha_2})$ and $n_0 > 5 + \max(\alpha_1^{-1}, \alpha_2^{-1})$, we obtain the domination of $\Psi_{n,l}(t_l)$ by an integrable function $\phi(t_l)$: By (13), we have

$$n \geq n_0, \quad \frac{|t_l|}{\sqrt{n}} \geq M \Longrightarrow |\Psi_{n,l}(t_l)| = \left|\varphi_l\left(\frac{t_l}{\sqrt{n}}\right)^n\right| \leq \left(\frac{\sqrt{n_0}c_l^{\frac{1}{\alpha_l}}}{|t_l|}\right)^{n_0\alpha_l}.$$

Moreover, by Smith,[8] (17), discussion right after Lemma 3 there exist a nonnegative constant ς_l such that

$$\frac{|t_l|}{\sqrt{n}} < M \Longrightarrow |\Psi_{n,l}(t_l)| = \left|\varphi_l\left(\frac{t_l}{\sqrt{n}}\right)^n\right| \leq e^{-\varsigma_l t_l^2}.$$

In order to dominate $\frac{\partial^5\Psi_n}{\partial t_k^5}(t_1, t_2)$, we observe the following.

- First, $\phi(t_l)$ is of the form $C\min(|t_l|^{-n_0\alpha_l}, e^{-\varsigma_l t_l^2})$. Since $|\varphi_k(\frac{t_k}{\sqrt{n}})^n| \leq \phi(t_k)$, then $|\varphi_k^{n-r}(\frac{t_k}{\sqrt{n}})| \leq \phi(t_k)^{\frac{n-r}{n}} \leq \phi_r(t_k)$, where for each $r \in [0, 5]$, ϕ_r is adequately chosen for large n. Moreover, by Lemma 4.3, we have

$$\lim_{n\to\infty}\varphi_k\left(\frac{t_k}{\sqrt{n}}\right)^n = \lim_{n\to\infty}\varphi_k^{n-r}\left(\frac{t_k}{\sqrt{n}}\right) = e^{-\frac{t_k^2}{2}}.$$

- Since $\varphi_k'(0) = 0$, then $|\varphi_k'(t_k)| = |\int_0^{t_k}\varphi_k''(u)du| \leq |t_k|$. Moreover, an expansion at the neighborhood of 0 of φ_k' shows that

$$\lim_{n\to\infty}\sqrt{n}\varphi_k'\left(\frac{t_k}{\sqrt{n}}\right) = -t_k.$$

Passing to the limit in (44), we retrieve

$$\lim_{n\to+\infty}\frac{\partial^5\Psi_n}{\partial t_k^5}(t_1, t_2) = -\left(t_k^5 - 10t_k^3 + 15t_k\right)e^{-\frac{1}{2}(t_1^2+t_2^2)}. \tag{45}$$

In its turn, $\hat{f}_{n,k}$ is integrable. By Fourier inversion formula, we obtain

$$f_{n,k}(t_1, t_2) = t_k^5 d_n(t_1, t_2) = \frac{-i}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\partial^5\Psi_n}{\partial t_k^5}(-u_1, -u_2)e^{iu_1t_1+iu_2t_2}du_1du_2.$$

Consequently, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}t_k^5 d_n(t_1, t_2) = \frac{i}{(2\pi)^2}\int_{\mathbb{R}^2}(-u_k^5 + 10u_k^3 - 15u_k)e^{iu_1t_1+iu_2t_2}e^{-\frac{1}{2}(u_1^2+u_2^2)}du_1du_2 = \frac{t_k^5 e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}.$$

Besides, this limit is uniform to (t_1, t_2), since $\frac{\partial^5\Psi_n}{\partial t_k^5}$ is dominated, independently on n, and since

$$\lim_{n\to\infty}\sup_{t_1,t_2}\left|t_k^5 d_n(t_1, t_2) - \frac{t_k^5 e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right| \leq \lim_{n\to\infty}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\frac{\partial^5\Psi_n}{\partial t_k^5}(u_1, u_2) + (u_k^5 - 10u_k^3 + 15u_k)e^{-\frac{1}{2}(u_1^2+u_2^2)}\right|du = 0.$$

This leads, immediately, to conclude

$$\lim_{n\to\infty}\sup_{t_1,t_2}\left| |t_k|^5 d_n(t_1,t_2) - \frac{|t_k|^5 e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right| = 0. \tag{46}$$

To prove that

$$\lim_{n\to\infty}\sup_{t_1,t_2}\left| |t_k|^m d_n(t_1,t_2) - \frac{|t_k|^m e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right| = 0, \text{ for all } m \in [0,5], \tag{47}$$

use

- in case $|t_k| \le 1$, use Corollary 4.4 together with

$$\left| |t_k|^m d_n(t_1,t_2) - |t_k|^m \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right| \le \left| d_n(t_1,t_2) - \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right|,$$

- in case $|t_k| > 1$, use (46) together with

$$\left| |t_k|^m d_n(t_1,t_2) - |t_k|^m \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right| \le |t_k|^5 \left| d_n(t_1,t_2) - \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\right|,$$

and conclude that is true. □

The following result will be used in the proof of Theorem 4.7.

Lemma 5.2 (Mode[17]). *Let h_n, f and g be given by (9) and (57), respectively, and z_1 be given by (53). Then,*

$$\sqrt{t_2}\sum_{-a\le z_1<a} h_n(t_1,t_2) = \frac{\sqrt{\lambda_2}}{\lambda_1}\int_{-a}^{a} f(x,t_1,t_2)\,dx + \frac{\sqrt{\lambda_2}}{\lambda_1}\int_{-a}^{a} g(y,t_1,t_2)\,dy.$$

Remark 5.3. As shown in Example 4.9, the constant $\sqrt{\lambda_2}$ is missing in the original statement of Mode.[17, Theorem 2.2]

Proof. Theorem 4.7. Due to the length of proof, we divide it into four steps and we set some notations. Let

$$X_k^* = \frac{X_k - \lambda_k}{\sigma_k},\ \ k = 1,\ 2,\ \mathbf{X}^* = (X_1^*, X_2^*)' \text{ and } \mathbf{X}_1^*,\ \dots\ , \mathbf{X}_n^* \text{ be independent copies of } \mathbf{X}^*. \tag{48}$$

Step 0: The goal of this step is to obtain a refinement of the asymptotic of $h_n(t_1,t_2)$ uniformly in (t_1,t_2). Since $\mathbf{X}^*$ is centered with identity covariance matrix, then by Corollary 4.4, the probability density function d_n of the random vector,

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbf{X}_k^* = \left(\frac{\sum_{k=1}^{n} X_{1,k} - n\lambda_1}{\sqrt{n}\sigma_1}, \frac{\sum_{k=1}^{n} X_{2,k} - n\lambda_2}{\sqrt{n}\sigma_2}\right),$$

satisfies

$$n\sigma_1\sigma_2 h_n(t_1,t_2) = d_n\left(\frac{t_1 - n\lambda_1}{\sqrt{n}\sigma_1}, \frac{t_2 - n\lambda_2}{\sqrt{n}\sigma_2}\right), \tag{49}$$

and as $n \to +\infty$ and uniformly in (t_1,t_2),

$$d_n(t_1,t_2) = \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\left(1 + \frac{\Lambda_1(t_1,t_2)}{\sqrt{n}}\frac{\Lambda_2(t_1,t_2)}{n}\right) + \Upsilon_n(t_1,t_2). \tag{50}$$

Using (49) and (50), we see that as n tends to infinity, uniformly in (t_1, t_2), we have

$$nh_n(t_1,t_2) = \frac{e^{-\frac{1}{2n}(t_1-n\lambda_1,t_2-n\lambda_2)'\Sigma^{-1}(t_1-n\lambda_1,t_2-n\lambda_2)}}{2\pi|\Sigma|^{\frac{1}{2}}}\left(1+\frac{\Lambda_1\left(\frac{t_1-n\lambda_1}{\sqrt{n}\sigma_1},\frac{t_2-n\lambda_2}{\sqrt{n}\sigma_2}\right)}{\sqrt{n}}+\frac{\Lambda_2\left(\frac{t_1-n\lambda_1}{\sqrt{n}\sigma_1},\frac{t_2-n\lambda_2}{\sqrt{n}\sigma_2}\right)}{n}\right)+\frac{1}{|\Sigma|^{\frac{1}{2}}}\Upsilon_n\left(\frac{t_1-n\lambda_1}{\sqrt{n}\sigma_1},\frac{t_2-n\lambda_2}{\sqrt{n}\sigma_2}\right).$$

Let us denote

$$V_{n,k,r}(t_1,t_2) = \left(\frac{t_k-n\lambda_k}{\sqrt{n}\sigma_k}\right)^r n\, h_n(t_1,t_2),\ W_{n,k,r}(t_1,t_2) = \left(\frac{t_k-n\lambda_k}{\sqrt{n}\sigma_k}\right)^r \frac{1}{2\pi\sqrt{|\Sigma|}}\exp\left(-\frac{1}{2n}(t_1-n\lambda_1,t_2-n\lambda_2)'\Sigma^{-1}(t_1-n\lambda_1,t_2-n\lambda_2)\right). \tag{51}$$

By Proposition 4.6, for all $r \in [0, 5]$ and $k = 1, 2$, we have

$$\lim_{n\to\infty}\left(V_{n,k,r}(t_1,t_2) - W_{n,k,r}(t_1,t_2)\right) = 0,\ \text{uniformly on}\ (t_1,t_2).$$

Then, for $r = 5,\ k = 1$, for all $\epsilon > 0$ small enough, there exists $n_0 \geq 1$ such that

$$h_n(t_1,t_2) \leq \frac{\epsilon}{(t_1-n\lambda_1)^5},\ \forall\, n \geq n_0. \tag{52}$$

Step 1: We prove that for n_0 given by (52),

$$\sqrt{t_2}\sum_{n=1}^{n_0-1} h_n(t_1,t_2) = o\left(\frac{1}{t_1}\right),\ t_1 \to \infty.$$

Indeed, under assumption (36), for all $n \geq 1$ and $\epsilon > 0$, there exists some $A_{n,\epsilon}$ such that

$$t_1 > A_{n,\epsilon} \Rightarrow t_1^{\frac{3}{2}} h_n(t_1,t_2) \leq \epsilon.$$

Observe that under the constraint $|\frac{t_1}{\lambda_1} - \frac{t_2}{\lambda_2}| \leq B$, the number $A_{n,\epsilon}$ can be chosen independently from t_2. Then,

$$t_1 > A^*_\epsilon := \max_{1\leq n\leq n_0-1} A_{n,\epsilon} \Rightarrow \sum_{n=1}^{n_0-1} h_n(t_1,t_2) \leq \frac{\epsilon\ n_0}{t_1^{\frac{3}{2}}}.$$

As before, under the constraint $|\frac{t_1}{\lambda_1} - \frac{t_2}{\lambda_2}| \leq B$ and $t_1, t_2 \to \infty$, there exists $C_1 \geq \max\limits_{1\leq n\leq n_0-1} A_{n,\epsilon}$ such that for

$$t_1,\ t_2 > C_1 \Rightarrow \sqrt{\frac{t_2}{t_1}} < \frac{3}{2} \Rightarrow \sqrt{t_2}\sum_{n=1}^{n_0-1} h_n(t_1,t_2) \leq \frac{\frac{3}{2}\epsilon\ n_0}{t_1}.$$

This means (5).

Step 2: Let us choose a an arbitrarily non-negative number such that

$$N := 2a\sqrt{\frac{t_1}{\lambda_1\lambda_2}} + 1 \in \mathbb{N}$$

and define the subdivision of the interval $[-a, a)$ by

$$[-a,a) = \bigcup_{j=1}^{N-1}[w_j,\ w_{j+1}),\quad \text{such that}\ \ w_{j+1} - w_j = \sqrt{\frac{\lambda_1\lambda_2}{t_1}},\ \forall\ 1 \leq j \leq N-1.$$

Define, for all $k = 1, 2$,

$$z_k := \sqrt{\frac{\lambda_k}{t_k}}\,(n\lambda_k - t_k),\ t_k > 0,\ k = 1,\ 2, \tag{53}$$

and observe that $1 + z_k(t_k\lambda_k)^{-\frac{1}{2}} = n\lambda_k/t_k > 0$. In this step, we aim to show that

$$\sqrt{t_2}\sum_{-a\le z_1<a} h_n(t_1,t_2) = K + \frac{C(t_1,t_2)}{t_1} + o\left(\frac{1}{t_1}\right),\ t_1 \to \infty.$$

From (53), and with F in (35), we have

$$z_2 = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{3}{2}}\sqrt{\frac{t_1}{t_2}}z_1 + \frac{\lambda_2^{\frac{3}{2}}}{\sqrt{t_2}}F(t_1,t_2) \text{ and } n = \frac{t_1}{\lambda_1} + z_1\frac{\sqrt{t_1}}{\lambda_1^{\frac{3}{2}}} = \frac{t_2}{\lambda_2} + z_2\frac{\sqrt{t_2}}{\lambda_2^{\frac{3}{2}}} = \frac{t_2}{\lambda_2} + \frac{\sqrt{t_1}}{\lambda_1^{\frac{3}{2}}}z_1 + F(t_1,t_2), \tag{54}$$

where F is given in (35). Take $r = 0$ in (51), and then denote, by $V(z_1,t_1,t_2)$ and $W(z_1,t_1,t_2)$, the functions obtained by injecting the number n in (54) into the functions in (51) and define the following functions:

$$V^*(z_1,t_1,t_2) := W(z_1,t_1,t_2)\prod_{k=1}^{2}\sqrt{\frac{1}{1+z_k(t_k\lambda_k)^{-\frac{1}{2}}}},$$
$$W^*(z_1,t_1,t_2) := [V(z_1,t_1,t_2) - W(z_1,t_1,t_2)]\prod_{k=1}^{2}\sqrt{\frac{1}{1+z_k(t_k\lambda_k)^{-\frac{1}{2}}}}.$$

Using (53), V^* and W^* can be expressed as

$$V^*(z_1,t_1,t_2) = W(z_1,t_1,t_2)\sqrt{\frac{1}{1+z_1(t_1\lambda_1)^{-\frac{1}{2}}}}\sqrt{\frac{t_2\lambda_2^{-1}}{t_2\lambda_2^{-1} + z_1\sqrt{t_1}\lambda_1^{-\frac{3}{2}} + F(t_1,t_2)}}, \tag{55}$$

$$W^*(z_1,t_1,t_2) = [V(z_1,t_1,t_2) - W(z_1,t_1,t_2)]\sqrt{\frac{1}{1+z_1(t_1\lambda_1)^{-\frac{1}{2}}}} \times \sqrt{\frac{t_2\lambda_2^{-1}}{t_2\lambda_2^{-1} + z_1\sqrt{t_1}\lambda_1^{-\frac{3}{2}} + F(t_1,t_2)}}. \tag{56}$$

Let f and g be two functions defined on $[-a,a)\times(0,\infty)^2$ by

$$f(y,t_1,t_2) = \sum_{j=1}^{N-1} V^*(y,t_1,t_2)\mathbf{1}_{[w_j,w_{j+1}]}(y),\ g(y,t_1,t_2) = \sum_{j=1}^{N-1} W^*(y,t_1,t_2)\mathbf{1}_{[w_j,w_{j+1}]}(y). \tag{57}$$

The function f can be written in the form

$$f(y,t_1,t_2) = \sqrt{\frac{1}{1+y(t_1\lambda_1)^{-\frac{1}{2}}}}\sqrt{\frac{t_2\lambda_2^{-1}}{t_2\lambda_2^{-1} + y\sqrt{t_1}\lambda_1^{-\frac{3}{2}} + F(t_1,t_2)}}\frac{R(y,t_1,t_2)}{2\pi|\Sigma|^{\frac{1}{2}}}, \tag{58}$$

where

$$R(y,t_1,t_2) = \exp\left\{-\frac{1}{2\left(t_1\lambda_1^{-1} + y\sqrt{t_1}\lambda_1^{-\frac{3}{2}}\right)}\left(\frac{y^2 t_1\lambda'\Sigma^{-1}\lambda}{\lambda_1^3} + \frac{\lambda_2^2 F(t_1,t_2)}{\sigma_2^2}\left(\frac{2\sqrt{t_1}y}{\lambda_1^{\frac{3}{2}}} + F(t_1,t_2)\right)\right)\right\}. \tag{59}$$

Since

$$\frac{\lambda_2}{t_2} = \frac{\lambda_1}{t_1} + o(t_1^{-\frac{3}{2}}),\ t_1,\ t_2 \to \infty,\ |F(t_1, t_2)| < B,$$

then

$$R(y, t_1, t_2) = e^{-y^2 \frac{\lambda' \Sigma^{-1} \lambda}{2\lambda_1^2}} \left[1 - \frac{y}{\sqrt{t_1}} \left(\frac{\lambda_2^2 F(t_1, t_2)}{\sigma_2^2 \sqrt{\lambda_1}} - \frac{y^2 \lambda' \Sigma^{-1} \lambda}{2\lambda_1^{\frac{5}{2}}} \right) + \frac{1}{t_1} \left\{ -\frac{\lambda_1 \lambda_2^2 F^2(t_1, t_2)}{2\sigma_2^2} + \frac{\lambda_2^2 y^2 F(t_1, t_2)}{\sigma_2^2 \lambda_1} \left(1 + \frac{\lambda_2^2 F(t_1, t_2)}{2\sigma_2^2} \right) \right.\right.$$
$$\left.\left. - \frac{y^4 \lambda' \Sigma^{-1} \lambda}{2\lambda_1^3} \left(1 + \frac{\lambda_2^2 F(t_1, t_2)}{\sigma_2^2} \right) + \frac{y^6 (\lambda' \Sigma^{-1} \lambda)^2}{8\lambda_1^5} \right\} \right] + o\left(\frac{1}{t_1}\right). \tag{60}$$

The first term on the right-hand side of Equation (58) can be expanded as

$$\sqrt{\frac{1}{1 + y(t_1 \lambda_1)^{-\frac{1}{2}}}} \sqrt{\frac{t_2 \lambda_2^{-1}}{t_2 \lambda_2^{-1} + y\sqrt{t_1} \lambda_1^{-\frac{3}{2}} + F(t_1, t_2)}} = 1 - \frac{y}{\sqrt{t_1 \lambda_1}} + \frac{1}{t_1} \left[\frac{y^2}{\lambda_1} - \frac{\lambda_1}{2} F(t_1, t_2) \right] + o\left(\frac{1}{t_1}\right). \tag{61}$$

Multiplying the terms in (59) and (60), we get

$$f(y, t_1, t_2) = \frac{e^{-y^2 \frac{\lambda' \Sigma^{-1} \lambda}{2\lambda_1^2}}}{2\pi \sqrt{|\Sigma|}} \left[1 + \frac{P_1(y, t_1, t_2)}{\sqrt{t_1}} + \frac{P_2(y, t_1, t_2)}{t_1} \right] + o\left(\frac{1}{t_1}\right), \tag{62}$$

where

$$P_1(y, t_1, t_2) = \frac{y}{\sqrt{\lambda_1}} \left(\frac{y^2 \lambda' \Sigma^{-1} \lambda}{2\lambda_1^2} - \frac{\lambda_2^2 F(t_1, t_2)}{\sigma_2^2} - 1 \right),$$

$$P_2(y, t_1, t_2) = -\frac{\lambda_1 F(t_1, t_2)}{2} - \frac{\lambda_1 \lambda_2^2 F^2(t_1, t_2)}{2\sigma_2^2} + \frac{y^2}{\lambda_1} \left[1 + \frac{\lambda_2^2 F(t_1, t_2)}{\sigma_2^2} (2 + \frac{\lambda_2^2 F(t_1, t_2)}{2\sigma_2^2}) \right]$$
$$- \frac{y^4 (\lambda' \Sigma^{-1} \lambda)}{2\lambda_1^3} [2 + \frac{\lambda_2^2 F(t_1, t_2)}{\sigma_2^2}] + \frac{y^6 (\lambda' \Sigma^{-1} \lambda)^2}{8\lambda_1^5}.$$

Let

$$G(y, t_1, t_2) = \left\{ t_1 f(y, t_1, t_2) - \frac{e^{-y^2 \frac{\lambda' \Sigma^{-1} \lambda}{2\lambda_1^2}}}{2\pi \sqrt{|\Sigma|}} \left[t_1 + P_1(y, t_1, t_2)\sqrt{t_1} + P_2(y, t_1, t_2) \right] \right\} \mathbf{1}_{\{|F(t_1, t_2)| \leq B\}}.$$

The function G may be written in the form

$$G(y, t_1, t_2) = \frac{t_1 e^{-\frac{y^2 (\lambda' \Sigma^{-1} \lambda)}{2\lambda_1^2}}}{2\pi \sqrt{|\Sigma|}} \left\{ \frac{\sqrt{t_1 \lambda_1^{-1} (t_1 \lambda_1^{-1} - F(t_1, t_2))}}{t_1 \lambda_1^{-1} + y\sqrt{t_1} \lambda_1^{-\frac{3}{2}}} R^*(y, t_1, t_2) - 1 - \frac{P_1(y, t_1, t_2)}{\sqrt{t_1}} - \frac{P_2(y, t_1, t_2)}{t_1} \right\} \mathbb{1}_{\{|F(t_1, t_2)| \leq B\}},$$

where

$$R^*(y, t_1, t_2) = e^{\frac{y^2 (\lambda' \Sigma^{-1} \lambda)}{2\lambda_1^2}} R(y, t_1, t_2). \tag{63}$$

Let us fix $a > 0$. We shall prove that for all $t_1 > t_0 := 4a^2\lambda_1^{-1}$, the function F is dominated by an integrable function on $[-a, a]$. For this end, we define

$$r_1^*(y,t_1,t_2) = \frac{y\lambda_2^2 F(t_1,t_2)}{\sigma_2^2\sqrt{\lambda_1}} - \frac{y^3(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^{\frac{5}{2}}}, \tag{64}$$

$$r_2^*(y,t_1,t_2) = \frac{\lambda_1\lambda_2^2F^2(t_1,t_2)}{2\sigma_2^2} - \frac{y^2\lambda_2^2F(t_1,t_2)}{\sigma_2^2\lambda_1} + \frac{y^4(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^3}, \tag{65}$$

$$q_1(y,t_1,t_2) = -\frac{y}{\sqrt{\lambda_1}},\ q_2(y,t_1,t_2) = \frac{y^2}{\lambda_1} - \frac{\lambda_1 F(t_1,t_2)}{2}, \tag{66}$$

and we notice that P_1 and P_2 can be expressed with r_1^*, r_2^*, q_1, and q_2 as follows:

$$P_1(y,t_1,t_2) = q_1(y,t_1,t_2) - r_1^*(y,t_1,t_2),\ \ P_2(y,t_1,t_2) = q_2(y,t_1,t_2) - q_1(y,t_1,t_2)r_1^*(y,t_1,t_2) + \frac{r_1^{*2}(y,t_1,t_2)}{2} - r_2^*(y,t_1,t_2).$$

Thus, G can be decomposed into

$$G(y,t_1,t_2) = \frac{e^{-\frac{y^2(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^2}}}{2\,\pi\sqrt{|\Sigma|}} \sum_{k=1}^{4} J_k(y,t_1,t_2)\ \mathbb{1}_{\{|F(t_1,t_2)|\leq B\}}, \tag{67}$$

with

$$J_1(y,t_1,t_2) = t_1\ R^*(y,t_1,t_2)\left[\frac{\sqrt{t_1\lambda_1^{-1}(t_1\lambda_1^{-1} - F(t_1,t_2))}}{t_1\lambda_1^{-1} + y\sqrt{t_1}\lambda_1^{-\frac{3}{2}}} - 1 - \frac{q_1(y,t_1,t_2)}{\sqrt{t_1}} - \frac{q_2(y,t_1,t_2)}{t_1}\right],$$

$$J_2(y,t_1,t_2) = t_1\ \left[R^*(y,t_1,t_2) - 1 + \frac{r_1^*(y,t_1,t_2)}{\sqrt{t_1}} + \frac{1}{t_1}\left(r_2^*(y,t_1,t_2) - \frac{r_1^{*2}(y,t_1,t_2)}{2}\right)\right],$$

$$J_3(y,t_1,t_2) = t_1\ \ (R^*(y,t_1,t_2) - 1)\left(\frac{q_1(y,t_1,t_2)}{\sqrt{t_1}} + \frac{q_2(y,t_1,t_2)}{t_1}\right),$$

$$J_4(y,t_1,t_2) = q_1(y,t_1,t_2)\ r_1^*(y,t_1,t_2).$$

Using the facts that for all $t_1 > t_0$,

$$R^*(y,t_1,t_2) \leq \exp\left\{\frac{\lambda_2^2 B}{\sigma_2^2} + \frac{(\lambda'\Sigma^{-1}\lambda)y^2}{2\lambda_1^2}\right\}$$

and

$$\frac{1}{(\sqrt{t_1} + y\lambda_1^{-\frac{1}{2}})^2} \leq \frac{\lambda_1}{a^2}, \qquad \frac{\sqrt{t_1}}{\sqrt{t_1} + y\lambda_1^{-\frac{1}{2}}} \leq 2, \qquad \frac{\sqrt{t_1}}{(\sqrt{t_1} + y\lambda_1^{-\frac{1}{2}})^2} \leq \frac{2\sqrt{\lambda_1}}{a},$$

we prove that all the functions $J_k(y,t_1,t_2),\ k = 1,\ \ldots\ ,4$, are dominated uniformly in t_1, t_2: Using the inequality

$$e^{-x} - 1 + x - \frac{x^2}{2} \leq x^2 e^{-x},$$

we obtain that

$$|J_1(y,t_1,t_2)| \leq \exp\left\{\frac{\lambda_2^2 B}{\sigma_2^2} + \frac{(\lambda'\Sigma^{-1}\lambda)y^2}{2\lambda_1^2}\right\}\left(2B\lambda_1 + \frac{|y|^3}{a\lambda_1} + \frac{\lambda_1 B}{2}\right)$$

$$|J_2(y,t_1,t_2)| \leq \left(1+\frac{|y|}{a}\right)^2\left(\frac{\lambda_2^2 B|y|}{\sigma_2^2\sqrt{\lambda_1}} + \frac{|y|^3(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^{\frac{5}{2}}} + \frac{\lambda_1^{\frac{3}{2}}\lambda_2^2 B}{4\sigma_2^2 a}\right)\exp\left\{\frac{\lambda_2^2 B}{\sigma_2^2} + \frac{(\lambda'\Sigma^{-1}\lambda)y^2}{2\lambda_1^2}\right\}$$
$$+\frac{\lambda_1\lambda_2^2 B^2}{2\sigma_2^2 a}\left(\frac{\lambda_2^2 B|y|}{\sigma_2^2} + \frac{|y|^3(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^2}\right) + \frac{3\lambda_1\lambda_2^2 B^2}{2\sigma_2^2} + \frac{\lambda_1^3\lambda_2^4 B^4}{8a^2\sigma_2^4} + \frac{3\lambda_2^2 B y^2}{\sigma_2^2\lambda_1} + \frac{3y^4(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^3}$$

$$|J_3(y,t_1,t_2)| \leq \left(\frac{|y|}{\sqrt{\lambda_1}} + \frac{\sqrt{\lambda_1}}{2a}\left(\frac{y^2}{\lambda_1} + \frac{\lambda_1 B}{2}\right)\right)\left(\frac{\lambda_1^{\frac{3}{2}}\lambda_2^2 B^2}{2a\sigma_2^2} + \frac{2\lambda_2^2 B|y|}{\sigma_2^2\sqrt{\lambda_1}} + \frac{|y|^3(\lambda'\Sigma^{-1}\lambda)}{\lambda_1^{\frac{5}{2}}}\right)\exp\left\{\frac{\lambda_2^2 B}{\sigma_2^2} + \frac{(\lambda'\Sigma^{-1}\lambda)y^2}{2\lambda_1^2}\right\}$$

$$|J_4(y,t_1,t_2)| \leq \frac{\lambda_2^2 B y^2}{\sigma_2^2\lambda_1} + \frac{y^4(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^3}.$$

Consequently, we conclude that, independently of t_1, t_2, $\sum_{k=1}^4 J_k(y,t_1,t_2)$ is dominated by an integrable function on $[-a,a)$, let denote this function by $H(y)$. We deduce then that for $t_1 > t_0$, the function G satisfies

$$G(y,t_1,t_2) \leq \frac{H(y)\, e^{\frac{y^2(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^2}}}{2\pi\sqrt{|\Sigma|}}. \tag{68}$$

Furthermore by (62), we have $G(y,t_1,t_2) \to 0$, as $||t_1,t_2|| \to +\infty$. By the dominated convergence theorem, we conclude that

$$\int_{-a}^{a} G(y,t_1,t_2)dy = o(1), \text{ if } ||t_1,t_2|| \to +\infty.$$

Thus, if $t_1,\ t_2 \to \infty$ with $|F(t_1,t_2)| \leq B$, then

$$\int_{-a}^{a} f(y,t_1,t_2)dy = \frac{1}{2\pi\sqrt{|\Sigma|}}\int_{-a}^{a} e^{-y^2\frac{\lambda'\Sigma^{-1}\lambda}{2\lambda_1^2}}\left[1 + \frac{P_1(y,t_1,t_2)}{\sqrt{t_1}} + \frac{P_2(y,t_1,t_2)}{t_1}\right]dy + o\left(\frac{1}{t_1}\right).$$

Let us denote by

$$K^*(\omega) = \frac{\lambda_1}{\sqrt{2\pi|\Sigma|(\lambda'\Sigma^{-1}\lambda)}}. \tag{69}$$

Using the fact that P_1 is an odd function and that

$$\int_{\mathbb{R}} y^{2n} e^{-y^2\frac{\lambda'\Sigma^{-1}\lambda}{2\lambda_1^2}}dy = \frac{\lambda_1^{2n+1}\sqrt{2\pi}}{(\lambda'\Sigma^{-1}\lambda)^{\frac{2n+1}{2}}}.1.3\ \ldots\ (2n-1), \text{ for all } n \geq 1,$$

we obtain, that for large enough a,

$$\int_{-a}^{a} f(y,t_1,t_2)dy = K^*(\omega)\left\{1 + \frac{\lambda_1}{2t_1}\left[-F(t_1,t_2)\left(1 + \frac{\lambda_2^2 F(t_1,t_2)}{\sigma_2^2}\right)\right.\right. \tag{70}$$

$$\left.\left. + \frac{1}{\lambda'\Sigma^{-1}\lambda}\left(-\frac{1}{4} + \frac{\lambda_2^2}{\sigma_2^2}F(t_1,t_2)\left(1 + \frac{\lambda_2^2}{\sigma_2^2}F(t_1,t_2)\right)\right)\right]\right\} + o\left(\frac{1}{t_1}\right). \tag{71}$$

Regarding the approximation of the function g (57), we base ourselves on Equation (49) and Theorem 4.4 to get

$$g(y,t_1,t_2)=\frac{e^{-\frac{y^2(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^2}}}{2\pi\sqrt{|\Sigma|}}\left[\frac{Q_1(y,t_1,t_2)}{\sqrt{t_1}}+\frac{Q_2(y,t_1,t_2)}{t_1}+\right]o\left(\frac{1}{t_1}\right),$$

where

$$\begin{aligned}
Q_1(y,t_1,t_2)=&\frac{3y}{\sqrt{\lambda_1}}\sum_{k=1}^{2}\frac{\lambda_k\mathbb{E}[(X_k^*)^3]}{\sigma_k}-\frac{y^3}{\lambda_1^{\frac{5}{2}}}\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[(X_k^*)^3]}{\sigma_k^3},\\
Q_2(y,t_1,t_2)=&\frac{3\lambda_1\lambda_2\mathbb{E}[(X_2^*)^3]F(t_1,t_2)}{\sigma_2}+\frac{\Theta\lambda_1}{12}\\
&+y^2\left[\frac{\lambda_1}{12}\sum_{k=1}^{2}\frac{\Xi_{k,1}}{\sigma_1^{2-k}}(\frac{\lambda_2}{\sigma_2\lambda_1})^k-\frac{3\lambda_2^3F(t_1,t_2)\mathbb{E}[(X_2^*)^3]}{\lambda_1\sigma_2^3}-3\left(2+\frac{\lambda_2^2F(t_1,t_2)}{\sigma_2^2}\right)\sum_{k=1}^{2}\frac{\lambda_k\mathbb{E}[(X_k^*)^3]}{\lambda_1\sigma_k}\right]\\
&+y^4\left[\left(3+\frac{\lambda_2^2F(t_1,t_2)}{\sigma_2^2}\right)\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[(X_k^*)^3]}{\lambda_1^3\sigma_k^3}+\frac{\lambda_1}{12}\sum_{k=1}^{4}\frac{\Xi_{k,2}}{\sigma_1^{4-k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^k+\frac{3(\lambda'\Sigma^{-1}\lambda)}{2\lambda_1^3}\sum_{k=1}^{2}\frac{\lambda_k\mathbb{E}[(X_k^*)^3]}{\sigma_k}\right]\\
&+\frac{y^6}{2}\left[\frac{\lambda_1}{6}\sum_{k=1}^{2}\frac{\Pi_k}{\sigma_1^{3k}}(\frac{\lambda_2}{\sigma_2\lambda_1})^{3(2-k)}-\frac{(\lambda'\Sigma^{-1}\lambda)}{\lambda_1^5}\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[(X_k^*)^3]}{\sigma_k^3}\right],
\end{aligned}$$

where X_k^* is given by (48). Due to Theorem 4.4, we deduce that g is dominated by an integrable function on $[-a,a)$; then, we conclude by the dominated convergence theorem that

$$\begin{aligned}
\int_{-a}^{a}g(y,t_1,t_2)dy=&\frac{\lambda_1\,K^*(\omega)}{2t_1}\left\{\frac{\lambda_2}{\sigma_2}-\frac{\lambda_1\,\mathbb{E}[(X_1^*)^3]}{\sigma_1(\lambda'\Sigma^{-1}\lambda)}\left(\frac{1}{2}+\frac{\lambda_2^2F(t_1,t_2)}{\sigma_2^2}\right)+\frac{\lambda_2\,\mathbb{E}[(X_2^*)^3]}{\sigma_2}\left(\frac{1}{2}+2\frac{\lambda_2^2F(t_1,t_2)}{\sigma_2^2}\right)\right.\\
&+\frac{\left(\frac{1}{2}+\frac{\lambda_2^2F(t_1,t_2)}{\sigma_2^2}\right)}{(\lambda'\Sigma^{-1}\lambda)^2}\sum_{k=1}^{2}\frac{\lambda_k^3\mathbb{E}[(X_k^*)^3]}{\sigma_k^3}+\frac{1}{36}\left[\Theta+\frac{\lambda_1^2}{\lambda'\Sigma^{-1}\lambda}\sum_{k=0}^{2}\frac{\Xi_{k,1}}{\sigma_1^{2-k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^k\right.\\
&\left.\left.+\frac{3\lambda_1^4}{(\lambda'\Sigma^{-1}\lambda)^2}\sum_{k=0}^{4}\frac{\Xi_{k,2}}{\sigma_1^{4-k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^k+\frac{15\lambda_1^6}{(\lambda'\Sigma^{-1}\lambda)^3}\sum_{k=0}^{4}\frac{\Pi_k}{\sigma_1^{3k}}\left(\frac{\lambda_2}{\sigma_2\lambda_1}\right)^{3(2-k)}\right]\right\}+o\left(\frac{1}{t_1}\right),
\end{aligned}\tag{72}$$

where all the constants are given in paragraph 2.4. Summing up (70) and (72), we obtain

$$\sqrt{t_2}\sum_{-a\le z_1<a}h_n(t_1,t_2)=K+\frac{C(t_1,t_2)}{t_1}+o\left(\frac{1}{t_1}\right),\tag{73}$$

where K is given by (14) and $C(t_1,t_2)$ is given by (35).

Step 3: For $|t_1|$ large, we have

$$\sqrt{t_2}\sum_{\substack{n\ge n_0\\ z_1\le -a\ \text{or}\ z_1>a}}h_n(t_1,t_2)=o\left(\frac{1}{t_1}\right).$$

Let $\alpha=t_1\lambda_1^{-1}+a\sqrt{t_1}\,\lambda_1^{-3/2}$, and z_1 is given in (53). By (52), we obtain for all $\epsilon>0$ that

$$\sqrt{t_2}\sum_{\substack{n\ge n_0\\ z_1\le -a\ \text{or}\ z_1>a}}h_n(t_1,t_2)\le 2\epsilon\sqrt{t_2}\sum_{\substack{n\ge n_0\\ n\ge\lceil\alpha\rceil+1}}\frac{1}{(n\lambda_1-t_1)^5}\le 2\epsilon\sqrt{t_2}\int_{\alpha-1}^{\infty}\frac{dv}{(v\lambda_1-t_1)^5}=\frac{\epsilon\sqrt{t_2}}{2\lambda_1(a\sqrt{t_1}\lambda_1^{-1/2}-\lambda_1)^4}.$$

Consequently,

$$\sqrt{t_2} \sum_{\substack{n \geq n_0 \\ z_1 \leq -a \text{ or } z_1 > a}} h_n(t_1, t_2) = o\left(\frac{1}{t_1}\right). \tag{74}$$

Using the last four steps, and Lemma 5.2, we conclude with the decomposition of the sum

$$\sum_{n=1}^{\infty} h_n(t_1, t_2) = \sum_{n=1}^{n_0-1} h_n(t_1, t_2) + \sum_{\substack{n \geq n_0 \\ -a \leq z_1 < a}} + \sum_{\substack{n \geq n_0 \\ z_1 \geq -a \text{ or } z_1 < a}} h_n(t_1, t_2).$$

□

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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