Moments and Centre of Mass

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Definition

Consider a thin plate T of density $\rho(x,y)$ which takes the form of a simple domain D of $\mathbb{R}^2$.
The mass of T is

$$M = \iint_D \rho(x,y)dxdy.$$

The moment of T with respect to the $x-$axis, respectively to the $y-$ axis are defined by:

$$M_x = \iint_D y\rho(x,y)dxdy, \qquad M_y = \iint_D x\rho(x,y)dxdy.$$

Definition

The center of mass or the centroid of T is

$$(\bar{x}, \bar{y}) = \frac{1}{M}(M_y, M_x).$$

In particular if $\rho = 1$, the mass M is the area of D and the center of mass of D is the point G of coordinates

$$(x_G, y_G) = \frac{1}{\mathrm{Area}(D)}\left(\iint_D x dx dy, \iint_D y dx dy\right).$$

Example

The center of mass of the disc D of center (a, b) and radius R is the point (a, b). Indeed, we have

$$\begin{aligned}
\iint_D x dx dy &= \int_{a-R}^{a+R} \int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{R^2-(x-a)^2}} x dy dx \\
&= \int_{a-R}^{a+R} 2x\sqrt{R^2-(x-a)^2} dx \\
&= \int_{-R}^{R} 2x\sqrt{R^2-x^2} dx + \int_{-R}^{R} 2a\sqrt{R^2-x^2} dx \\
&= a\,\mathrm{Aire}(D).
\end{aligned}$$

Example

Consider the triangle with vertices $(0, 0)$, $(0, 2)$, $(3, 0)$ and with density $\rho(x, y) = xy$. Find the total mass and center of mass.

$$\begin{aligned} M &= \iint_T \rho(x, y)dxdy = \int_0^3 \int_0^{2-x} xy\, dy\, dx \\ &= \frac{9}{8}. \end{aligned}$$

$$\begin{aligned} \overline{x} &= \frac{8}{9} \iint_T x\rho(x, y)dxdy = \frac{8}{9} \int_0^3 \int_0^{2-x} x^2 y\, dy\, dx \\ &= \frac{8}{5}. \end{aligned}$$

$$\begin{aligned}\overline{y} &= \frac{8}{9}\iint_T y\rho(x,y)dxdy = \frac{8}{9}\int_0^3\int_0^{2-x} xy^2\,dy\,dx\\ &= \frac{3}{10}.\end{aligned}$$

Example

The center of mass of the disc D of center (a, b) and radius R is the point (a, b). Indeed, we have

$$\begin{aligned}
\iint_D x dx dy &= \int_{a-R}^{a+R} \int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{R^2-(x-a)^2}} x dy dx \\
&= \int_{a-R}^{a+R} 2x\sqrt{R^2-(x-a)^2} dx \\
&= \int_{-R}^{R} 2x\sqrt{R^2-x^2} dx + \int_{-R}^{R} 2a\sqrt{R^2-x^2} dx \\
&= a \,\text{Aire}(D)
\end{aligned}$$

Example

Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the $x-$axis between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.
The density is constant, we can take $\sigma(x, y) = 1$. Then $\bar{x} = 0$.
$M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} dy\, dx = 2$, $M_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} y\, dy\, dx = \dfrac{\pi}{4}$ and
$M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} x\, dy\, dx = 0$. So $\bar{x} = 0$ and $\bar{y} = \frac{\pi}{8}$.

Example

Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant and has density $k(x^2 + y^2)$.

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} k(x^2 + y^2)\, dy\, dx = k \int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3}\, dx = \frac{k\pi}{8}.$$

$$M_x = k \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \sin\theta\, dr\, d\theta = \frac{k}{5}, \quad M_y = k \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos\theta\, dr\, d\theta = \frac{k}{5}.$$

Then , $\bar{x} = \bar{y} = \dfrac{8}{5\pi}$.

Moment of Inertia of a Lamina

Definition

Consider a thin plate T of density $\rho(x, y)$ which takes the form of a simple domain D of $\mathbb{R}^2$.

The moment about the $x-$axis is $I_x = \iint_D y^2 \rho(x, y) dxdy$.

The moment about the $y-$axis is $I_y = \iint_D x^2 \rho(x, y) dxdy$.

The moment about the origin is

$I_0 = \iint_D (x^2 + y^2)\rho(x, y) dxdy = I_x + I_y$.

Centres of Mass of Solid

Definition

Consider a solid D of density $\rho(x, y, z)$.
The mass of D is

$$M = \iiint_D \rho(x, y, z)dxdydz.$$

The moment of D with respect to the $xy-$plane, respectively to the yz and the $xz-$planes are defined by:

$$M_{xy} = \iiint_D z\rho(x, y, z)dxdyz, \qquad M_{yz} = \iiint_D x\rho(x, y, z)dxdyz,$$

$$M_{xz} = \iiint_D y\rho(x, y, z)dxdyz.$$

Definition

The center of mass or the centroid of D is

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}(M_{yz}, M_{xz}, M_{xy}).$$

In particular if $\rho = 1$, the mass M is the volume of D and the center of mass of D is the point G of coordinates

$$(x_G, y_G, z_G) = \frac{1}{\mathrm{Volume}(D)}\left(\iiint_D x dxdydz, \iiint_D y dxdydz, \iiint_D z dxdydz\right).$$

Definition

Consider a solid D of density $\rho(x, y, z)$.
The moment about the $x-$axis is
$I_x = \iint_D (y^2 + z^2)\rho(x, y, z)dxdydz$.
The moment about the $y-$axis is
$I_y = \iint_D (x^2 + z^2)\rho(x, y, z)dxdydz$.
The moment about the $z-$axis is
$I_z = \iint_D (x^2 + y^2)\rho(x, y, z)dxdydz$.
The moment about the origin is $I_0 = I_x + I_y + I_z$.

Example

Suppose the density of an object is given by $\rho(x, y, z) = xz$, and the object occupies the tetrahedron with corners $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$. Find the mass and center of mass of the object.

As usual, the mass is the integral of density over the region:

$$\begin{aligned}
M &= \int_0^1 \int_x^1 \int_0^{y-x} xz \, dz \, dy \, dx \\
&= \int_0^1 \int_x^1 \frac{x(y-x)^2}{2} \, dy \, dx \\
&= \frac{1}{2} \int_0^1 \frac{x(1-x)^3}{3} \, dx \\
&= \frac{1}{6} \int_0^1 x - 3x^2 + 3x^3 - x^4 \, dx = \frac{1}{120}.
\end{aligned}$$

We compute moments as before, except now there is a third moment:

$$
\begin{aligned}
M_{xy} &= \int_0^1 \int_x^1 \int_0^{y-x} xz^2 \, dz \, dy \, dx = \frac{1}{360}, \\
M_{xz} &= \int_0^1 \int_x^1 \int_0^{y-x} xyz \, dz \, dy \, dx = \frac{1}{144}, \\
M_{yz} &= \int_0^1 \int_x^1 \int_0^{y-x} x^2 z \, dz \, dy \, dx = \frac{1}{360}.
\end{aligned}
$$

Finally, the coordinates of the center of mass are $\bar{x} = \dfrac{M_{yz}}{M} = \dfrac{1}{3}$, $\bar{y} = \dfrac{M_{xz}}{M} = \dfrac{5}{6}$, and $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{1}{3}$.

Example

Find the mass and center of mass of the solid with density $\rho(x, y, z)$ and the given shape.

1. $\rho(x, y, z) = 4$, solid bounded by $z = x^2 + y^2$ and $z = 4$
2. $\rho(x, y, z) = 2 + x$, solid bounded by $z = x^2 + y^2$ and $z = 4$
3. $\rho(x, y, z) = 10 + x$, tetrahedron bounded by $x + 3y + z = 6$ and the coordinate planes
4. $\rho(x, y, z) = 1 + x$, tetrahedron bound by $2x + y + 4z = 4$ and the coordinate planes.

Exercise 1 :

Find the center of mass of a two-dimensional plate that occupies the square $[0,1] \times [0,1]$ and has density function xy.

Exercise 2 :

Find the center of mass of a two-dimensional plate that occupies the triangle $0 \leq x \leq 1$, $0 \leq y \leq x$, and has density function xy.

Exercise 3 :
Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at (0,0) and has density function y.

Exercise 4 :
Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0, 0)$ and has density function x^2.

Exercise 5 :
Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 2, y = x$, and $y = 2x$ and has density function $2x$.

Exercise 6 :
Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 0$, $y = x$, and $2x + y = 6$ and has density function x^2.

Exercise 7 :
Find the center of mass of a two-dimensional plate that occupies the region enclosed by the parabolas $x = y^2$, $y = x^2$ and has density function $\sqrt{x}$.

Exercise 8 :

Find the centroid of the area in the first quadrant bounded by $x^2 - 8y + 4 = 0, x^2 = 4y$, and $x = 0$. (Recall that the centroid is the center of mass when the density is 1 everywhere.)

Exercise 9 :

Find the centroid of one loop of the three-leaf rose $r = \cos(3\theta)$. (Recall that the centroid is the center of mass when the density is 1 everywhere, and that the mass in this case is the same as the area, which was the subject of exercise 11 in section 15.2.) The computations of the integrals for the moments M_x and M_y are elementary but quite long; Sage can help.

Exercise 10 :

Find the center of mass of a two dimensional object that occupies the region $0 \leq x \leq \pi, 0 \leq y \leq \sin x$, with density $\sigma = 1$.

Exercise 11 :

A two-dimensional object has shape given by $r = 1 + \cos\theta$ and density $\sigma(r,\theta) = 2 + \cos\theta$. Set up the three integrals required to compute the center of mass.

Exercise 12 :

A two-dimensional object has shape given by $r = \cos\theta$ and density $\sigma(r,\theta) = r + 1$. Set up the three integrals required to compute the center of mass.

Exercise 13 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

Exercise 14 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

Exercise 15 :
An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.

Exercise 16 :

Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

Exercise 17 :

Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

Exercise 18 :
An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.

Exercise 19 :

An object occupies the volume of the pyramid with corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 2)$ and has density $x^2 + y^2$ at (x, y, z). Find the center of mass.

Exercise 20 :

Consider the triangular wedge D that is in the first octant, bounded by the planes $\dfrac{y}{7} + \dfrac{z}{5} = 1$ and $x = 12$. In the yz plane, the wedge forms a triangle that passes through the points $(0, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 5)$. Set up integral formulas that would give the centroid $(\bar{x}, \bar{y}, \bar{z})$ of D. Actually compute the integrals for $\bar{y}$. Then state $\bar{x}$ and $\bar{z}$ by using symmetry arguments.