Final Exam, Semester I, 1446

Department of Mathematics, College of Science, KSU

Course: Math 481 Marks: 40 Duration: 3 Hours

immediate

Question 1 [3+2+4 points]

1. Prove that if f is continuous on $[a, b]$, then $f \in \mathcal{R}(a, b)$. (See the book by Al-Gwaiz.)

2. Provide an example of a function that is Riemann integrable but not continuous.

   Answer: An example of a function that is Riemann integrable but not continuous is the function, defined as:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,\, 1/2], \\ 0 & \text{if } x \in (1/2,\, 1]. \end{cases}$$

   Why is this function Riemann integrable?

   - The function is bounded because for all x, $f(x) \in [0, 1]$.
   - The set of discontinuities, $\{1/2\}$, has measure zero. Specifically, it is a finite set, and finite sets have Lebesgue measure zero.
   - According to the *Lebesgue Criterion for Riemann Integrability*, a function is Riemann integrable if it is bounded and its set of discontinuities has measure zero.

3. Suppose (f_n) is a sequence of Riemann integrable functions that converges uniformly on $[a, b]$ to a function f. Prove that f is Riemann integrable on $[a, b]$ and that:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

   (See the book by Al-Gwaiz.)

Question 2 [4+4 points]

1. The functions f_n on $[-1, 1]$ are defined by $f_n(x) = \dfrac{x}{1 + n^2x^2}$. Show that the pointwise limit of f_n is differentiable, but the equality $f'(x) = \lim_{n\to\infty} f_n'(x)$ does not hold for all $x \in [-1, 1]$. **Solution: Step 1: Pointwise limit of** $f_n(x)$

- For $x = 0$:

$$f_n(0) = 0.$$

- For $x \neq 0$, as $n \to \infty$:

$$f_n(x) = \frac{x}{1 + n^2 x^2} \to 0.$$

Thus, the pointwise limit is the constant function:

$$f(x) = 0, \quad \forall x \in [-1, 1].$$

Step 2: Differentiability of $f(x)$ Since $f(x) = 0$, its derivative is:

$$f'(x) = 0, \quad \forall x \in [-1, 1].$$

Step 3: Derivative of $f_n(x)$ Using the quotient rule:

$$f_n'(x) = \frac{1 - n^2 x^2}{(1 + n^2 x^2)^2}.$$

Step 4: Limit of $f_n'(x)$

- For $x = 0$:

$$f_n'(0) = 1, \quad \lim_{n \to \infty} f_n'(0) = 1.$$

- For $x \neq 0$, as $n \to \infty$:

$$f_n'(x) \to 0 \quad \text{because } n^2 x^2 \to \infty.$$

Step 5: Comparison

- $f'(x) = 0$ for all x.
- $\lim_{n \to \infty} f_n'(x) = 1$ at $x = 0$, so $f'(0) \neq \lim_{n \to \infty} f_n'(0)$.
- For $x \neq 0$, $f'(x) = \lim_{n \to \infty} f_n'(x) = 0$.

Conclusion: The pointwise limit $f(x)$ is differentiable, but $f'(x) \neq \lim_{n \to \infty} f_n'(x)$ at $x = 0$.

2. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n^2}{n^3}$$

converges uniformly on any bounded interval, but does not converge uniformly on $\mathbb{R}$.

Solution:

Since the series is even in the sense that its terms involve x^2, it suffices to show that the series converges uniformly on the interval $[0, a]$, where $a > 0$. To do this, we will apply Dirichlet's Test for Uniform Convergence.

Dirichlet's Test states that if we have a series of the form

$$\sum_{n=1}^{\infty} u_n(x)v_n(x),$$

where:

- $u_n(x)$ is a monotonically decreasing sequence that tends to zero uniformly on $[0, a]$,
- $v_n(x)$ is a sequence such that the partial sums $S_N = \sum_{n=1}^{N} v_n$ are bounded for all N, and for all $x \in [0, a]$,

then the series $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ converges uniformly on $[0, a]$.

We choose:

$$v_n(x) = (-1)^n \quad \text{and} \quad u_n(x) = \frac{x^2 + n^2}{n^3}.$$

We now verify the two conditions of Dirichlet's Test.

Condition 1: Monotonicity of $u_n(x)$ and Uniform Limit Condition

-u_n is monotonically decreasing on $[0, a]$
- We now check the behavior of $u_n(x)$. Since

$$u_n(x) = \frac{x^2 + n^2}{n^3} \leq \frac{a^2 + n^2}{n^3} \quad \text{for} \quad x \in [0, a],$$

it follows that $u_n(x) \to 0$ uniformly as $n \to \infty$, since

$$\frac{a^2 + n^2}{n^3} = \frac{a^2}{n^3} + \frac{1}{n}.$$

Both terms go to zero as $n \to \infty$, and the convergence is uniform for $x \in [0, a]$.

Condition 2: Boundedness of Partial Sums of $v_n(x)$

We check the partial sums of $v_n(x)$. We know that $v_n(x) = (-1)^n$, so the partial sums are alternating sums of ± 1. Thus, for any N,

$$\left| \sum_{n=1}^{N} v_n(x) \right| \leq 2,$$

which is bounded. Therefore, both conditions of Dirichlet's test are satisfied, and we conclude that the series converges uniformly on the interval $[0, a]$.

Next, we show that the series does not converge uniformly on $\mathbb{R}$. To do this, we will show that the general term does not converge uniformly to zero. Indeed,

$$\sup_{x \in \mathbb{R}} \left| \frac{x^2 + n^2}{n^3} \right| \geq \sup_{x \in \mathbb{R}} \frac{x^2}{n^3} \geq 1 \quad \text{for} \quad x = n^{3/2}.$$

Thus, the general term does not uniformly converge to zero on $\mathbb{R}$, and the series does not converge uniformly on $\mathbb{R}$.

Question 3 [2+2+4 points]

(a) Define a measurable set and a measurable function.(See the Book of AlGwaiz)

(b) Given E such that $m^*(E) = 0$, prove that E is a measurable set.

Solution: To prove that E is a measurable set, we need to show that for every set $A \subset \mathbb{R}$, the following equality holds:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

First, we observe that the inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

is trivially satisfied due to the subadditivity of the outer measure. This follows because:

$$A = (A \cap E) \cup (A \cap E^c).$$

Now, we prove the reverse inequality. Since $A \cap E \subseteq E$, and given that $m^*(E) = 0$, we have:

$$m^*(A \cap E) \leq m^*(E) = 0.$$

Therefore, we conclude:

$$m^*(A \cap E) = 0.$$

Next, we observe that $A \cap E^c \subseteq A$, so by the monotonicity of the outer measure, we have:

$$m^*(A \cap E^c) \leq m^*(A).$$

Thus, combining these results, we obtain:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

This shows that E is a measurable set, as required.

(c) Consider the function:

$$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x > 1, \\ 0, & \text{if } x \leq 1. \end{cases}$$

i. Is this function Riemann integrable on the interval $[1, a]$ for all $a > 1$?
Solution:
A function $f(x)$ is Riemann integrable on $[1, a]$ for $a > 1$ if it is bounded and its discontinuities form a set of measure zero. If $f(x)$ is continuous or has only a finite number of discontinuities on $[1, a]$, it is Riemann integrable.

ii. Show that this function is Lebesgue integrable on the interval $[1, \infty)$ and

$$\int_{[1,\infty)} f(x)\, dm = 1.$$

Solution: The function $f(x)$ is non-negative and measurable. To show that $f(x)$ is Lebesgue integrable on $[1, \infty)$, we check whether

$$\int_{[1,\infty)} |f(x)|\, dm < \infty.$$

Let $f_n(x) = \chi_{[1,n]}(x)\frac{1}{x^2}$, where $\chi_{[1,n]}$ is the characteristic function of the interval $[1, n]$. Clearly, $f_n(x) \nearrow f(x)$ as $n \to \infty$ by the Monotone Convergence Theorem, we can write:

$$\int_{[1,\infty)} f(x)\, dm = \lim_{n\to\infty} \int_{[1,n]} f(x)\, dm.$$

Since $f_n(x) = \frac{1}{x^2}$ on the interval $[1, n]$, we have:

$$\int_{[1,n]} \frac{1}{x^2}\, dm = \int_1^n \frac{1}{x^2}\, dx.$$

The integral of $\frac{1}{x^2}$ is straightforward to compute:

$$\int_1^n \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^n = 1 - \frac{1}{n}.$$

Taking the limit as $n \to \infty$, we get:

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

Since the integral is finite, $f(x) = \frac{1}{x^2}$ is Lebesgue integrable on $[1, \infty)$, and we conclude:

$$\int_{[1,\infty)} f(x)\, dm = 1.$$

This completes the proof.

Question 4 [2+3 points]

(a) Provide an example of a function that is Lebesgue integrable but not Riemann integrable.

(b) Evaluate the limit of the following integral:

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1 + n^2x^2}\, dx.$$

Solution:
Define:

$$I_n = \int_0^1 \frac{nx}{1 + n^2x^2}\, dx.$$

For each $x \in [0,1]$, the integrand $f_n(x) = \frac{nx}{1+n^2x^2}$ satisfies $f_n(x) \to 0$ as $n \to \infty$. Moreover, $f_n(x) \leq \frac{1}{2}$, since the maximum value of $f_n(x)$ occurs at $x = \frac{1}{n}$, where $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$. By the *Bounded Convergence Theorem*, we can exchange the limit and the integral, yielding:

$$\lim_{n\to\infty} I_n = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

Thus:

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx = 0.$$

Question 5 [3+3+2+2 points]

(a) Prove that

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx = 1.$$

Solution:

Define:

$$I_n = \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx.$$

Step 1: Majorization For all $x \geq 0$, we have:

$$\left(1+\frac{x}{n}\right)^n \leq e^x.$$

Thus, the integrand satisfies:

$$\left(1+\frac{x}{n}\right)^n e^{-2x} \leq e^x e^{-2x} = e^{-x}.$$

The function e^{-x} is integrable on $[0,\infty)$ because:

$$\int_0^\infty e^{-x}\,dx = 1.$$

Step 2: Pointwise Convergence For fixed $x \in [0,\infty)$, as $n \to \infty$, we have:

$$\left(1+\frac{x}{n}\right)^n \to e^x.$$

Thus, the integrand $\left(1+\frac{x}{n}\right)^n e^{-2x}$ converges pointwise to:

$$e^x e^{-2x} = e^{-x}.$$

Step 3: Applying the Dominated Convergence Theorem Since the integrand is dominated by e^{-x}, which is integrable on $[0, \infty)$, we can apply the Dominated Convergence Theorem. Hence:

$$\lim_{n\to\infty} I_n = \int_0^\infty e^{-x}\,dx = 1.$$

Conclusion:

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx = 1.$$

(b) If $p, q > 0$, prove that

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx = \sum_{n=0}^\infty \frac{(-1)^n}{p+nq}.$$

$$\frac{nx}{1+n^2x^2} = nx\sum_{k=0}^\infty (-1)^k (n^2x^2)^k.$$

Using the series expansion:

$$\frac{x^{p-1}}{1+x^q} = \sum_{k=0}^n (-1)^k x^{p-1+kq} + \frac{(-1)^{n+1}x^{p-1+q(n+1)}}{1+x^q},$$

we integrate over $[0, 1]$:

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx = \int_0^1 \sum_{k=0}^n (-1)^k x^{p-1+kq}\,dx + \int_0^1 \frac{(-1)^{n+1}x^{p-1+q(n+1)}}{1+x^q}\,dx.$$

For the first term:

$$\int_0^1 \sum_{k=0}^n (-1)^k x^{p-1+kq}\,dx = \sum_{k=0}^n (-1)^k \int_0^1 x^{p-1+kq}\,dx = \sum_{k=0}^n \frac{(-1)^k}{p+kq}.$$

For the remainder term:

$$R_n = \int_0^1 \frac{(-1)^{n+1}x^{p-1+q(n+1)}}{1+x^q}\,dx,$$

we start by bounding the integrand. Since $\frac{1}{1+x^q} \leq 1$ for all $x \in [0, 1]$, it follows that:

$$\left|\frac{(-1)^{n+1}x^{p-1+q(n+1)}}{1+x^q}\right| \leq 1, \quad \text{for all } x \in [0, 1].$$

The sequence:

$$\frac{(-1)^{n+1}x^{p-1+q(n+1)}}{1+x^q}$$

converges pointwise to 0 almost everywhere on $[0, 1]$ as $n \to \infty$.

By the *Dominated Convergence Theorem*, the integral of the remainder term goes to 0 as $n \to \infty$. Therefore:

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq}.$$

Then, we can conclude the following:

i.

$$\log 2 = \int_0^1 \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n}.$$

ii.

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Then, we can conclude the following:

i.

$$\log 2 = \int_0^1 \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n}.$$

Explanation: Setting $p = 1$ and $q = 1$ in the derived formula:

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq},$$

we get:

$$\int_0^1 \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n}.$$

The integral $\int_0^1 \frac{1}{1+x}\,dx$ is the standard representation of $\log 2$, so:

$$\log 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n}.$$

ii.

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Explanation: Setting $p = 1$ and $q = 2$ in the derived formula:

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq},$$

we get:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

The integral $\int_0^1 \frac{1}{1+x^2}\,dx$ is the standard representation of $\arctan(1)$, and since $\arctan(1) = \frac{\pi}{4}$, we conclude:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$