Model Solution: Second Mid-term, Semester II, 1447

Department of Mathematics, College of Science, KSU

Course: Math 209 Maximum Marks: 25 Duration: 1.5 Hours

Question 1 [4+4+4 points]

(1) Find the interval and radius of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n, \qquad \sum_{n=1}^{\infty} \frac{1}{4^n n^2} (x+1)^n$$

Solution.

(i) For

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n,$$

let

$$a_n = \frac{2^n x^n}{n!}.$$

By the ratio test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right| = \frac{2|x|}{n+1} \to 0 \qquad (n \to \infty).$$

Hence the series converges for every real x.

Therefore,

$$R = \infty, \qquad \text{interval of convergence } = (-\infty, \infty).$$

(ii) For

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^2} (x+1)^n,$$

let

$$a_n = \frac{(x+1)^n}{4^n n^2}.$$

Using the root test,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x+1|}{4} \sqrt[n]{\frac{1}{n^2}} = \frac{|x+1|}{4}.$$

Hence the series converges when

$$\frac{|x+1|}{4} < 1 \quad \Longleftrightarrow \quad |x+1| < 4.$$

So the radius of convergence is

$$R = 4.$$

The open interval is

$$-5 < x < 3.$$

At $x = -5$,

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^2}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges absolutely.

At $x = 3$,

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^2}(4)^n = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Hence the interval of convergence is

$$[-5, 3].$$

(2) Find the power series expansion centered at 0 for the functions:

$$e^{2x}, \qquad \frac{1}{1+x}$$

Solution.

(i) The Maclaurin series of e^u is

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}.$$

Putting $u = 2x$, we obtain

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n.$$

Thus

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots.$$

(ii) Using the geometric series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \qquad |r| < 1,$$

and taking $r = -x$, we get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1.$$

Therefore

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \qquad |x| < 1.$$

(3) Use the result from part (2) to find the power series for:

$$\log\left(\sqrt{1+x}\right), \qquad \int_0^x \frac{e^{2t}-1}{t}\,dt$$

Solution.

(i) Since

$$\log\left(\sqrt{1+x}\right) = \frac{1}{2}\log(1+x),$$

we differentiate first:

$$\frac{d}{dx}\log\left(\sqrt{1+x}\right) = \frac{1}{2(1+x)}.$$

From part (2),

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1.$$

Hence

$$\frac{1}{2(1+x)} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n x^n.$$

Integrating term by term from 0 to x,

$$\log\left(\sqrt{1+x}\right) - \log(\sqrt{1}) = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt.$$

Since $\log(\sqrt{1}) = 0$, we get

$$\log\left(\sqrt{1+x}\right) = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

Therefore

$$\log\left(\sqrt{1+x}\right) = \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right), \qquad |x| < 1.$$

(ii) We use

$$e^{2t} = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!}.$$

Then

$$e^{2t} - 1 = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}.$$

Dividing by t,

$$\frac{e^{2t} - 1}{t} = \sum_{n=1}^{\infty} \frac{2^n t^{n-1}}{n!}.$$

Integrating from 0 to x,

$$\int_0^x \frac{e^{2t} - 1}{t} \, dt = \sum_{n=1}^{\infty} \frac{2^n}{n!} \int_0^x t^{n-1} \, dt = \sum_{n=1}^{\infty} \frac{2^n x^n}{n \, n!}.$$

Hence

$$\int_0^x \frac{e^{2t} - 1}{t} \, dt = \sum_{n=1}^{\infty} \frac{2^n}{n \, n!} x^n.$$

Question 2 **[3+3 points]**

Consider the periodic function defined by:

$$f(x) = 1 + x^2, \qquad x \in [-\pi, \pi],$$

and extended periodically with $f(x + 2\pi) = f(x)$.

(a) Find the Fourier series representation of $f(x)$.

Solution.

Since $f(x) = 1 + x^2$ is even, its Fourier series contains only cosine terms:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \qquad b_n = 0.$$

First,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x^2) \, dx = \frac{2}{\pi} \int_0^{\pi} (1 + x^2) \, dx.$$

Therefore

$$a_0 = \frac{2}{\pi} \left[x + \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left(\pi + \frac{\pi^3}{3} \right) = 2 + \frac{2\pi^2}{3}.$$

Hence

$$\frac{a_0}{2} = 1 + \frac{\pi^2}{3}.$$

Next,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x^2) \cos(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} (1 + x^2) \cos(nx) \, dx.$$

Split the integral:

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos(nx) \, dx + \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) \, dx.$$

But

$$\int_0^\pi \cos(nx)\,dx = \left[\frac{\sin(nx)}{n}\right]_0^\pi = 0.$$

So

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)\,dx.$$

From the standard computation for x^2,

$$\int_0^\pi x^2\cos(nx)\,dx = \frac{2\pi(-1)^n}{n^2}.$$

Hence

$$a_n = \frac{2}{\pi}\cdot\frac{2\pi(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}.$$

Therefore the Fourier series is

$$1 + x^2 = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \qquad -\pi < x < \pi.$$

(b) Deduce that:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution.

Put $x = 0$ in the Fourier series:

$$1 = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(0).$$

Since $\cos(0) = 1$, we get

$$1 = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

Therefore

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Question 3 [3+3 points]

Find the Fourier integral representation of the function:

$$f(x) = \begin{cases} 3, & |x| \le 2, \\ 0, & |x| > 2. \end{cases}$$

Then deduce the result:

$$\int_0^\infty \frac{\sin(2x)}{x}\,dx = \frac{\pi}{2}.$$

Solution.

The function f is even, so its Fourier integral representation is

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega,$$

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,dt.$$

Since

$$f(t) = 3 \quad \text{for } 0 \le t \le 2, \qquad f(t) = 0 \quad \text{for } t > 2,$$

we get

$$A(\omega) = \frac{2}{\pi}\int_0^2 3\cos(\omega t)\,dt = \frac{6}{\pi}\int_0^2 \cos(\omega t)\,dt.$$

Thus

$$A(\omega) = \frac{6}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_0^2 = \frac{6}{\pi}\cdot\frac{\sin(2\omega)}{\omega}.$$

Therefore

$$f(x) = \frac{6}{\pi}\int_0^\infty \frac{\sin(2\omega)}{\omega}\cos(\omega x)\,d\omega.$$

Hence the Fourier integral representation is

$$\boxed{f(x) = \frac{6}{\pi}\int_0^\infty \frac{\sin(2\omega)}{\omega}\cos(\omega x)\,d\omega}$$

with the usual half-sum value at the jump points. Thus

$$f(x) = \begin{cases} 3, & |x| < 2, \\ \frac{3}{2}, & |x| = 2, \\ 0, & |x| > 2. \end{cases}$$

To deduce the integral, put $x = 0$. Then $f(0) = 3$, so

$$3 = \frac{6}{\pi}\int_0^\infty \frac{\sin(2\omega)}{\omega}\,d\omega.$$

Hence

$$\int_0^\infty \frac{\sin(2\omega)}{\omega}\,d\omega = \frac{\pi}{2}.$$

Therefore

$$\boxed{\int_0^\infty \frac{\sin(2x)}{x}\,dx = \frac{\pi}{2}.}$$