$\boxed{Q_1}$

a) $X \sim \text{uniform}(0, 4000)$

$$E[X] = \int_0^{4000} x \frac{1}{4000}\, dx = \frac{1}{4000} \frac{x^2}{2} \Big|_{x=0}^{4000}$$

$$= 2000.$$

b) $$E[X^2] = \int_0^{4000} x^2 \frac{1}{4000}\, dx$$

$$= \frac{1}{4000} \frac{x^3}{3} \Big|_{x=0}^{4000} = \frac{1}{\cancel{4000}} \frac{4000^{\cancel{3}}}{3} = \frac{4000^2}{3}$$

$$= 5,333,333.33$$

$$\Rightarrow Var[X] = E[X^2] - (E[X])^2$$

$$= 5,333,333.33 - 2000^2 = 1,333,333.33$$

$$\Rightarrow SD[X] = 1,154.701$$

c) $$E[Y] = \int_d^{4000} (x - d)\, f_x(x)\, dx$$

$$= \int_{400}^{4000} (x - 400) \frac{1}{4000}\, dx$$

$$= \frac{1}{4000} \frac{(x-400)^2}{2} \Big|_{400}^{4000}$$

$$= \frac{1}{4000} \frac{(3600)^2}{2} = 1,620$$

d) If k is the amount of loss not covered by the policy, then

$$K = \begin{cases} X & \text{if } 0 \leq X \leq 400 \\ 400 & \text{if } 400 \leq X < 4000 \end{cases}$$

$$\Rightarrow E[K] = \int_0^{400} x \frac{1}{4000} dx + \int_{400}^{4000} 400 \frac{1}{4000} dx$$

$$= \frac{1}{4000} \frac{x^2}{2} \Big|_{x=0}^{400} + 400 \Pr(400 \leq X \leq 4000)$$

$$= \frac{400^2}{2 * 4000} + 400 * \frac{9}{10}$$

$$= 20 + 360 = 380$$

$\boxed{Q_2}$

a) $P_{GG} = P_r\{X_{n+1} = G \mid X_n = G\} = 0.4$

$P_{GD} = P_r\{X_{n+1} = D \mid X_n = G\} = 0.6$

$P_{DD} = P_r\{X_{n+1} = D \mid X_n = D\} = 0.3$

$P_{DG} = P_r\{X_{n+1} = G \mid X_n = D\} = 0.7$

$P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G, X_6 = D \mid X_1 = G\} = ?$

conditional property

$= P_r\{X_6 = D \mid X_2 = G, X_3 = G, X_4 = G, X_5 = G, X_1 = G\} \cdot P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G \mid X_1 = G\}$

Markov property

$= \underbrace{P_r\{X_6 = D \mid X_5 = G\}} \cdot \underline{P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G \mid X_1 = G\}}$

$= P_{GD} \cdot$ repeat the step.

$\left\{ \underline{P_r\{X_5 = G \mid X_2 = G, X_3 = G, X_4 = G, X_1 = G\}} \cdot P_r\left\{ \begin{matrix} X_2 = G \\ X_3 = G \\ X_4 = G \end{matrix} \middle| X_1 = G \right\} \right\}$

Markov Property

$P_r\{X_5 = G \mid X_4 = G\}$

$= P_{GD} \cdot P_{GG} \cdot \ldots$ and so on continous.

we get:

$= P_{GD} \cdot P_{GG}^4 = 0.6\,(0.4)^4 = 0.01536$

Q2

b) $P_r(X_5=2 \mid X_3=1)$

$$= P_{12}^2 = 0.72$$

$$P^2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} & 0.72 \\ & \end{bmatrix} \end{array}$$

Q3

a)

	(S,S)	(S,C)	(C,S)	(C,C)
(S,S)	0.7	0.3	0	0
(S,C)	0	0	0.4	0.6
(C,S)	0.5	0.5	0	0
(C,C)	0	0	0.2	0.8
	$\downarrow$ π_0	$\downarrow$ π_1	$\downarrow$ π_2	$\downarrow$ π_3

Q3 b) limiting distributi is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$\begin{cases} 0.7\pi_0 + 0.5\pi_2 = \pi_0 \Rightarrow \pi_2 = 0.6\pi_0 \\ 0.3\pi_0 + 0.5\pi_2 = \pi_1 \Rightarrow \pi_1 = 0.6\pi_0 \\ 0.6\pi_1 + 0.8\pi_3 = \pi_3 \Rightarrow \pi_3 = 1.8\pi_0 \\ \& \ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$\Rightarrow \quad \pi_0 = \frac{1}{4} = 0.25$

$\Rightarrow \pi = (0.25, 0.15, 0.15, 0.45)$

$\therefore$ The long run fraction of days in which it is cloudy is $\pi_2 + \pi_3 = 0.15 + 0.45 = 0.60$

& The long run fraction of days in which it is Sunny is $\pi_0 + \pi_1 = 0.25 + 0.15 = 0.40$

Q4

a) $\lambda = 2$, $X(t)$ = # messages that arrives at the telegraph office at anytime.

$$P_r\{\underset{s+t}{X(12)} - \underset{s}{X(8)} = \underset{k}{1}\} = \frac{(2*4)^1 e^{-2*4}}{1!}$$

$$= 8 \cdot e^{-8}$$

$\boxed{Q_4}$ b) $P_r\{X(2) = 5\}$

$$P_r\{X(2) - X(0) = 5\}$$

$$\mu = \int_0^2 \lambda(u)\,du$$

$$= \int_0^1 (2t+1)\,dt + \int_1^2 3\,dt$$

$$= \cancel{2}\frac{t^2}{\cancel{2}} + t\Big|_0^1 + 3t\Big|_1^2$$

$$= 2 + 3 = 5$$

$$\therefore P_r\{X(2) = 5\} = P_r\{X(2) - X(0) = \underset{k}{5}\}$$

$$= \frac{\mu^k e^{-\mu}}{k!} = \frac{5^5 e^{-5}}{5!}$$

$\boxed{Q5}$ a) (i) $P_r\{X_2=1\} = P_r\{X_2=1 \mid X_0=1\} \cdot P_r\{X_0=1\}$

$$= P_{11}^2 \cdot P_1 \qquad (P_1 = P_r\{X_0=1\}=1)$$

$$= P_{11}^2 = 0.22$$

$$P^2 = \begin{bmatrix} 0.49 & 0.28 & 0.23 \\ 0.43 & 0.22 & 0.35 \\ 0.47 & 0.20 & 0.33 \end{bmatrix}$$

(ii) To find $\pi = (\pi_0, \pi_1, \pi_2)$

Solve :

$$\begin{cases} 4\pi_0 - 3\pi_1 - 4\pi_2 = 0 \\ 3\pi_0 - 7\pi_1 + \pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

by Cramer's Rule

$\Delta = -66$, $\Delta_0 = -31$, $\Delta_1 = -16$, $\Delta_2 = -19$

$\therefore \pi_0 = \dfrac{\Delta_0}{\Delta} = \dfrac{31}{66}$, $\pi_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{16}{66}$, $\pi_2 = \dfrac{19}{66}$

(iii) $\pi_2 = \dfrac{19}{66} \simeq 0.2879$

$\boxed{Q6}$

a) $P_r\{B(4) > a \mid B(0) = 1\} = 0.1587$

$\Rightarrow \quad 1 - Pr\{B(4) \leq a \mid B(0) = 1\} = 0.1587$

$\Rightarrow \quad 1 - \Phi\left(\frac{a-1}{\sqrt{4}}\right) = 0.1587$

$\Rightarrow \quad \Phi\left(\frac{a-1}{2}\right) = 0.8413$

$\Rightarrow \quad \frac{a-1}{2} = 1$ using Normal tables

$\Rightarrow a - 1 = 2$

$\Rightarrow \boxed{a = 3}$

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b) $X(t) = cB\left(\frac{t}{c^2}\right)$, $c>0$, is a BM.

$\boxed{1}$ show $X(t) - X(s) \sim N(0, t-s)$:

$$X(t) - X(s)$$
$$= cB\left(\frac{t}{c^2}\right) - c\,B\left(\frac{s}{c^2}\right)$$
$$= c\left[B\left(\frac{t}{c^2}\right) - B\left(\frac{s}{c^2}\right)\right]$$
$$\sim c\ N\left(0, \frac{t}{c^2} - \frac{s}{c^2}\right)$$
$$\sim N\left(c.0\,;\ \cancel{c^2}.\frac{t-s}{\cancel{c^2}}\right)$$

من خواص BM

$\Rightarrow X(t) - X(s) \sim N(0, t-s)$ ✓

$\boxed{2}$ Show $Cov[X(t), X(s)] = \min(s,t)$:

Case ① $0 \leq s \leq t$

Since $B(t)$ is BM , $Cov[B(s), B(t)] = \min(s,t)$ $\forall s,t>0$

as $0 \leq s \leq t$

$\Rightarrow 0 \leq \frac{s}{c^2} \leq \frac{t}{c^2}$

So $Cov\left[B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right] = \frac{s}{c^2}$

from $Cov(aX,aY) = a^2 cov(X,Y)$

$$\Rightarrow c^2 . cov\left[B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right] = s$$
$$\Rightarrow Cov\left[c.B\left(\frac{s}{c^2}\right), cB\left(\frac{t}{c^2}\right)\right] = s$$
$$= Cov\left[X(s), X(t)\right] = s = \min(s,t)$$

Similarly, Case ② $0 \leq t \leq s$.

$\boxed{3}$ $E[X(t)] = E\left[cB\left(\frac{t}{c^2}\right)\right] = 0$

as $N\left(0, \frac{t}{c^2}\right)$

$\boxed{1}$, $\boxed{2}$ & $\boxed{3}$ $\Rightarrow X(t)$ is a BM.

c) $X(t) = e^{uB(t) - \frac{u^2}{2}t}$ is a martingale.

$\boxed{1}$ $E\,|X(t)| < \infty$?

$$E\left| e^{uB(t) - \frac{u^2}{2}t} \right| = E\left[e^{uB(t)} \cdot e^{-\frac{u^2}{2}t} \right]$$

$$= e^{-\frac{u^2}{2}t}\, E\left[e^{uB(t)} \right]$$

as $B(t) \sim N(0,t) \Rightarrow uB(t) \sim N(0, u^2 t)$

$$E\left[e^{uB(t)}\right] = e^{0 + \frac{u^2 t}{2}} = e^{\frac{u^2}{2}t}$$

$B(t) - B(s) \sim N(0, t-s)$

$$\Rightarrow E\left[e^{u(B(t)-B(s))} \right] = e^{\frac{u^2}{2}(t-s)}$$

$$\therefore E\,|X(t)| = e^{-\frac{u^2}{2}t} \cdot e^{\frac{u^2}{2}t} = e^{0} = 1 < \infty$$

✓

$\boxed{2}$ for $0 \le s \le t$, $E[X(t) \mid \mathcal{F}_s] = X(s)$?

$$\text{L.H.S.} = E\left[e^{uB(t) - \frac{u^2}{2}t} \mid \mathcal{F}_s \right]$$

$$= e^{-\frac{u^2}{2}t}\, E\left[e^{uB(t)} \mid \mathcal{F}_s \right]$$

$$= e^{-\frac{u^2}{2}t}\, E\left[e^{u(B(t) - B(s) + B(s))} \mid \mathcal{F}_s \right]$$

$$= e^{-\frac{u^2}{2}t}\, E\left[e^{u(B(t) - B(s))} \cdot \underbrace{e^{uB(s)}}_{\text{determined by } \mathcal{F}_s} \mid \mathcal{F}_s \right]$$

$$= e^{uB(s)}\, e^{-\frac{u^2}{2}t}\, E\left[e^{u(\underbrace{B(t) - B(s)}_{\text{independent of } \mathcal{F}_s})} \mid \mathcal{F}_s \right]$$

$$= e^{uB(s)}\, e^{-\frac{u^2}{2}t}\, \underbrace{E\left[e^{u(B(t)-B(s))} \right]}_{e^{\frac{u^2}{2}(t-s)}}$$

$$= e^{uB(s)}\, e^{-\frac{u^2}{2}t}\, e^{\frac{u^2}{2}(t-s)} = e^{uB(s) - s} = X(s) = \text{R.H.S.}$$

$\boxed{1}$ & $\boxed{2}$ $\Rightarrow X(t)$ is a martingale.

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