Model Solution: Second Mid-term, Semester II, 1446

Department of Mathematics, College of Science, KSU

Course: Math 209 Maximum Marks: 25 Duration: 1.5 Hours

Question 1 [4+4+4 points]

(1) Find the interval and radius of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{1}{n!} x^n, \qquad \sum_{n=1}^{\infty} \frac{1}{3^n n^2} (x-2)^n$$

Solution.

(i) $\sum_{n=1}^{\infty} \frac{1}{n!} x^n$

Let

$$a_n = \frac{x^n}{n!}.$$

Using the ratio test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \quad (n \to \infty).$$

Since the limit is $0 < 1$ for every $x \in \mathbb{R}$, the series converges for all real x.

Therefore,

$$R = \infty, \qquad \text{interval of convergence } = (-\infty, \infty).$$

(ii) $\sum_{n=1}^{\infty} \frac{1}{3^n n^2} (x-2)^n$

Let

$$a_n = \frac{(x-2)^n}{3^n n^2}.$$

Using the root test,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-2|}{3} \sqrt[n]{\frac{1}{n^2}} = \frac{|x-2|}{3}.$$

Hence the series converges when

$$\frac{|x-2|}{3} < 1 \quad \Longleftrightarrow \quad |x-2| < 3.$$

So the radius of convergence is

$$R = 3.$$

The corresponding open interval is

$$-1 < x < 5.$$

We now test the endpoints.

At $x = -1$:

$$\sum_{n=1}^{\infty} \frac{1}{3^n n^2}(-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges absolutely since $\sum 1/n^2$ converges.

At $x = 5$:

$$\sum_{n=1}^{\infty} \frac{1}{3^n n^2}(3)^n = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Therefore, the interval of convergence is

$$[-1, 5].$$

(2) Find the power series expansion centered at 0 for the functions:

$$e^x, \qquad \frac{x}{1+x^2}$$

Solution.

(i) e^x

The Maclaurin series of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad x \in \mathbb{R}.$$

(ii) $\dfrac{x}{1+x^2}$

We use the geometric series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \qquad |r| < 1.$$

Put $r = -x^2$. Then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad |x| < 1.$$

Multiplying by x,

$$\frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} = x - x^3 + x^5 - x^7 + \cdots, \qquad |x| < 1.$$

(3) Use the result from part (2) to find the power series for:

$$\log\left(\sqrt{1+x^2}\right), \qquad \int_0^x \frac{e^t - 1}{t}\, dt$$

Solution.

(i) $\log(\sqrt{1+x^2})$

Since

$$\log(\sqrt{1+x^2}) = \frac{1}{2}\log(1+x^2),$$

it is convenient to differentiate first:

$$\frac{d}{dx}\log(\sqrt{1+x^2}) = \frac{x}{1+x^2}.$$

From part (2),

$$\frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}, \qquad |x| < 1.$$

Integrating term by term from 0 to x,

$$\log(\sqrt{1+x^2}) - \log(\sqrt{1+0}) = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n+1}\, dt.$$

Since $\log(\sqrt{1}) = 0$, we get

$$\log(\sqrt{1+x^2}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+2}.$$

Hence

$$\log(\sqrt{1+x^2}) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \cdots, \qquad |x| < 1.$$

(ii) $\displaystyle\int_0^x \frac{e^t - 1}{t}\, dt$

From the expansion of e^t,

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots .$$

Therefore,

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!},$$

and for $t \neq 0$,

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}.$$

Integrating term by term from 0 to x,

$$\int_0^x \frac{e^t - 1}{t}\, dt = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_0^x t^n\, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n+1)!}.$$

Reindexing with $m = n + 1$,

$$\int_0^x \frac{e^t - 1}{t}\, dt = \sum_{m=1}^{\infty} \frac{x^m}{m \cdot m!}.$$

So

$$\int_0^x \frac{e^t - 1}{t}\, dt = x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdots .$$

Question 2 [3+3 points]

Consider the periodic function defined by:

$$f(x) = x^2, \qquad x \in [-\pi, \pi],$$

and extended periodically with $f(x + 2\pi) = f(x)$.

(a) Find the Fourier series representation of $f(x)$.

Solution.

Since $f(x) = x^2$ is an even function, its Fourier series contains only cosine terms:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx, \qquad b_n = 0.$$

First,

$$a_0 = \frac{1}{\pi} \cdot 2 \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

Hence

$$\frac{a_0}{2} = \frac{\pi^2}{3}.$$

Next,

$$a_n = \frac{2}{\pi} \int_0^\pi x^2 \cos(nx)\, dx.$$

Let

$$I_n = \int_0^\pi x^2 \cos(nx)\, dx.$$

We integrate by parts twice.

First integration by parts:

$$u = x^2, \qquad dv = \cos(nx)\, dx,$$

so

$$du = 2x\, dx, \qquad v = \frac{\sin(nx)}{n}.$$

Then

$$I_n = \left[\frac{x^2 \sin(nx)}{n}\right]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx)\, dx.$$

Since $\sin(n\pi) = 0$, this becomes

$$I_n = -\frac{2}{n} \int_0^\pi x \sin(nx)\, dx.$$

Now let

$$J_n = \int_0^\pi x \sin(nx)\, dx.$$

Again integrate by parts:

$$u = x, \qquad dv = \sin(nx)\, dx,$$

so

$$du = dx, \qquad v = -\frac{\cos(nx)}{n}.$$

Therefore,

$$J_n = \left[-\frac{x \cos(nx)}{n}\right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx)\, dx.$$

Since

$$\int_0^\pi \cos(nx)\, dx = \left[\frac{\sin(nx)}{n}\right]_0^\pi = 0,$$

we get

$$J_n = -\frac{\pi \cos(n\pi)}{n} = -\frac{\pi(-1)^n}{n}.$$

Hence

$$I_n = -\frac{2}{n} J_n = -\frac{2}{n}\left(-\frac{\pi(-1)^n}{n}\right) = \frac{2\pi(-1)^n}{n^2}.$$

Therefore,

$$a_n = \frac{2}{\pi} I_n = \frac{2}{\pi} \cdot \frac{2\pi(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}.$$

So the Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx), \qquad -\pi < x < \pi.$$

(b) Deduce that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution.

Use the Fourier series obtained in part (a):

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Put $x = \pi$. Since $\cos(n\pi) = (-1)^n$, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n.$$

Thus

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence

$$4\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{2\pi^2}{3} = \frac{\pi^2}{6}.$$

Question 3 [3+3 points]

Find the Fourier integral representation of the function:

$$f(x) = \begin{cases} 2, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Then deduce the result:

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Solution.

The function f is even. For an even function, the Fourier integral representation is

$$f(x) = \int_0^\infty a(\omega)\cos(\omega x)\,d\omega,$$

where

$$a(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,dt.$$

Since $f(t) = 2$ for $0 \le t \le 1$ and $f(t) = 0$ for $t > 1$, we have

$$a(\omega) = \frac{2}{\pi}\int_0^1 2\cos(\omega t)\,dt = \frac{4}{\pi}\int_0^1 \cos(\omega t)\,dt.$$

Thus

$$a(\omega) = \frac{4}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_0^1 = \frac{4}{\pi}\cdot\frac{\sin\omega}{\omega}.$$

Hence the Fourier integral representation is

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos(\omega x)\,d\omega.$$

Therefore,

$$f(x) = \begin{cases} 2, & |x| < 1, \\ 1, & |x| = 1, \\ 0, & |x| > 1, \end{cases}$$

in the Fourier integral sense. So we may write

$$\boxed{f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos(\omega x)\,d\omega}$$

with the usual half-sum value at the jump points $x = \pm 1$.

To deduce $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, put $x = 0$. Then $f(0) = 2$, so

$$2 = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega.$$

Hence

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{4}\cdot 2 = \frac{\pi}{2}.$$

Therefore,

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.}$$