Model Solution: Midterm Exam 1, Semester II, 1447

Department of Mathematics, College of Science, KSU

Course: Math 481

Question 1

Determine whether each function is Riemann integrable on $[0,1]$. If it is integrable, compute its integral.

1.

$$f(x) = \begin{cases} x, & x \in [0, \frac{1}{2}), \\ 1-2x, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Solution:

The function is continuous everywhere except possibly at $x = \frac{1}{2}$. We check:

$$\lim_{x \to (1/2)^-} f(x) = \frac{1}{2}, \qquad f\left(\frac{1}{2}\right) = 1 - 2 \cdot \frac{1}{2} = 0.$$

So there is one jump discontinuity at $x = \frac{1}{2}$.

A bounded function with only finitely many discontinuities is Riemann integrable. Hence f is Riemann integrable on $[0,1]$.

Now compute:

$$\int_0^1 f(x)\,dx = \int_0^{1/2} x\,dx + \int_{1/2}^1 (1-2x)\,dx.$$

First,

$$\int_0^{1/2} x\,dx = \left[\frac{x^2}{2}\right]_0^{1/2} = \frac{1}{8}.$$

Second,

$$\int_{1/2}^1 (1-2x)\,dx = \left[x - x^2\right]_{1/2}^1 = (1-1) - \left(\frac{1}{2} - \frac{1}{4}\right) = -\frac{1}{4}.$$

Therefore,

$$\int_0^1 f(x)\,dx = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}.$$

$$\boxed{f \text{ is Riemann integrable and } \int_0^1 f(x)\,dx = -\frac{1}{8}.}$$

2.
$$f(x) = \begin{cases} \frac{1}{2}, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Solution:

This function is bounded on $[0, 1]$. However, every interval contains both rational and irrational numbers, so on every subinterval:

$$\sup f = \frac{1}{2}, \qquad \inf f = 0.$$

Hence for any partition P of $[0, 1]$,

$$U(f, P) = \frac{1}{2}, \qquad L(f, P) = 0.$$

Since the upper and lower sums are never equal, the function is not Riemann integrable.

$$\boxed{f \text{ is not Riemann integrable on } [0, 1].}$$

Question 2

Express the following limits as definite integrals and evaluate them.

1. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n + 2k}$

Solution:

Rewrite:

$$\sum_{k=1}^{n} \frac{1}{n + 2k} = \sum_{k=1}^{n} \frac{1}{n\,(1 + 2k/n)} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + 2k/n}.$$

This is a Riemann sum for the function

$$f(x) = \frac{1}{1 + 2x}$$

on $[0, 1]$. Therefore,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n + 2k} = \int_0^1 \frac{1}{1 + 2x}\,dx.$$

Now evaluate:

$$\int_0^1 \frac{1}{1 + 2x}\,dx = \frac{1}{2} \ln(1 + 2x)\Big|_0^1 = \frac{1}{2} \ln 3.$$

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n + 2k} = \int_0^1 \frac{1}{1 + 2x}\,dx = \frac{1}{2} \ln 3.}$$

2. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+3k^2}$

   Solution:

   Rewrite:

$$\frac{k}{n^2+3k^2} = \frac{(k/n)}{n\left(1+3(k/n)^2\right)}.$$

   Thus

$$\sum_{k=1}^{n} \frac{k}{n^2+3k^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{k/n}{1+3(k/n)^2}.$$

   This is a Riemann sum for

$$f(x) = \frac{x}{1+3x^2}$$

   on $[0,1]$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+3k^2} = \int_0^1 \frac{x}{1+3x^2}\,dx.$$

   Now evaluate:

$$\int_0^1 \frac{x}{1+3x^2}\,dx.$$

   Let $u = 1+3x^2$, then $du = 6x\,dx$, so

$$\int_0^1 \frac{x}{1+3x^2}\,dx = \frac{1}{6}\int_1^4 \frac{1}{u}\,du = \frac{1}{6}\ln 4.$$

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+3k^2} = \int_0^1 \frac{x}{1+3x^2}\,dx = \frac{1}{6}\ln 4.}$$

Question 3

Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n^2 x, & x \in [0, 1/n], \\ n^2(x - 2/n), & x \in (1/n, 2/n], \\ 0, & x \in (2/n, 1]. \end{cases}$$

1. Show that $f_n(x) \to 0$ for all $x \in [0,1]$.

   Solution:

   Fix $x \in [0,1]$.

   If $x = 0$, then $f_n(0) = 0$ for all n, so clearly $f_n(0) \to 0$.

   Now let $x > 0$. Choose N such that $2/N < x$. Then for all $n \geq N$, we have $2/n < x$, so $x \in (2/n, 1]$. Hence

$$f_n(x) = 0 \qquad \text{for all } n \geq N.$$

Therefore $f_n(x) \to 0$.

So for every $x \in [0, 1]$,

$$\boxed{f_n(x) \to 0.}$$

2. Compute $\int_0^1 f_n(x)\,dx$.

   Solution:

   Since $f_n(x) = 0$ on $(2/n, 1]$,

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} n^2(x - 2/n)\,dx.$$

   First,

$$\int_0^{1/n} n^2 x\,dx = n^2 \left[\frac{x^2}{2}\right]_0^{1/n} = n^2 \cdot \frac{1}{2n^2} = \frac{1}{2}.$$

   Second,

$$\int_{1/n}^{2/n} n^2(x - 2/n)\,dx = n^2 \left[\frac{x^2}{2} - \frac{2x}{n}\right]_{1/n}^{2/n}.$$

   At $x = 2/n$:

$$\frac{(2/n)^2}{2} - \frac{2(2/n)}{n} = \frac{2}{n^2} - \frac{4}{n^2} = -\frac{2}{n^2}.$$

   At $x = 1/n$:

$$\frac{(1/n)^2}{2} - \frac{2(1/n)}{n} = \frac{1}{2n^2} - \frac{2}{n^2} = -\frac{3}{2n^2}.$$

   So

$$\int_{1/n}^{2/n} n^2(x - 2/n)\,dx = n^2 \left(-\frac{2}{n^2} + \frac{3}{2n^2}\right) = -\frac{1}{2}.$$

   Hence

$$\int_0^1 f_n(x)\,dx = \frac{1}{2} - \frac{1}{2} = 0.$$

$$\boxed{\int_0^1 f_n(x)\,dx = 0 \quad \text{for all } n.}$$

3. Determine whether $f_n \to 0$ uniformly on $[0, 1]$.

   Solution:

   Uniform convergence to 0 would require

$$\sup_{x \in [0,1]} |f_n(x)| \to 0.$$

Now

$$f_n\left(\frac{1}{n}\right) = n^2 \cdot \frac{1}{n} = n,$$

so

$$\sup_{x\in[0,1]} |f_n(x)| \geq n.$$

In fact, the maximum absolute value is n, since the function reaches n at $x = 1/n$ and $-n$ at $x = 2/n$.

Thus

$$\sup_{x\in[0,1]} |f_n(x)| = n \not\to 0.$$

Therefore the convergence is not uniform.

$$\boxed{f_n \not\to 0 \text{ uniformly on } [0,1].}$$

Question 4

For each sequence of functions below, determine whether the convergence is **pointwise** and/or **uniform** on $[0,1]$.

1. $f_n(x) = \dfrac{x}{nx+1}$

   Solution:

   Fix $x \in [0,1]$.

   If $x = 0$, then

   $$f_n(0) = 0.$$

   If $x > 0$, then

   $$f_n(x) = \frac{x}{nx+1} \to 0 \qquad (n \to \infty),$$

   since the denominator tends to ∞.

   So $f_n(x) \to 0$ pointwise on $[0,1]$.

   For uniform convergence, compute

   $$\sup_{x\in[0,1]} \left|\frac{x}{nx+1}\right|.$$

   Since the function is increasing in x for $x \geq 0$, the maximum occurs at $x = 1$:

   $$\sup_{x\in[0,1]} \frac{x}{nx+1} = \frac{1}{n+1} \to 0.$$

   Hence the convergence is uniform.

   $$\boxed{f_n \to 0 \text{ pointwise and uniformly on } [0,1].}$$

2. $f_n(x) = \dfrac{nx^3}{1+nx}$

Solution:

Fix $x \in [0,1]$.

If $x = 0$, then $f_n(0) = 0$.

If $x > 0$, divide numerator and denominator by n:

$$f_n(x) = \frac{x^3}{x + 1/n} \to \frac{x^3}{x} = x^2.$$

Thus the pointwise limit is

$$f(x) = \begin{cases} 0, & x = 0, \\ x^2, & x > 0. \end{cases}$$

Since x^2 also gives 0 at $x = 0$, we may write simply

$$f(x) = x^2.$$

So $f_n(x) \to x^2$ pointwise on $[0,1]$.

To test uniform convergence, consider

$$|f_n(x) - x^2| = \left| \frac{nx^3}{1+nx} - x^2 \right| = x^2 \left| \frac{nx}{1+nx} - 1 \right| = \frac{x^2}{1+nx}.$$

Define

$$g_n(x) = \frac{x^2}{1+nx}.$$

For $x \in [0,1]$, this is increasing, so its maximum occurs at $x = 1$:

$$\sup_{x\in[0,1]} |f_n(x) - x^2| = \frac{1}{n+1} \to 0.$$

Therefore the convergence is uniform.

$$\boxed{f_n \to x^2 \text{ pointwise and uniformly on } [0,1].}$$

3. $f_n(x) = x^n(1-x)$

Solution:

Fix $x \in [0,1]$.

If $0 \le x < 1$, then $x^n \to 0$, so

$$f_n(x) = x^n(1-x) \to 0.$$

If $x = 1$, then

$$f_n(1) = 1^n(1-1) = 0.$$

Hence $f_n(x) \to 0$ pointwise on $[0,1]$.

Now check uniform convergence. Since $f_n(x) \geq 0$ on $[0,1]$,

$$\sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} x^n(1-x).$$

Let

$$g_n(x) = x^n(1-x).$$

Differentiate:

$$g_n'(x) = nx^{n-1}(1-x) - x^n = x^{n-1}(n-(n+1)x).$$

Thus the critical point in $(0,1)$ is

$$x = \frac{n}{n+1}.$$

So the maximum is attained there:

$$g_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}.$$

Since

$$0 < \left(\frac{n}{n+1}\right)^n < 1,$$

we get

$$0 \leq \sup_{x\in[0,1]} |f_n(x)| = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \leq \frac{1}{n+1} \to 0.$$

Therefore $f_n \to 0$ uniformly on $[0,1]$.

$$\boxed{f_n \to 0 \text{ pointwise and uniformly on } [0,1].}$$

End of Model Solution