King Saud University, College of Science
Department of Mathematics
Midterm Exam 1447 – 1448 H - (2025 – 2026 G)

Second Midterm Exam of Math 431.
Allotted time: one and a half hours

Question 1

Consider the following two sequences of integers:

$$D_1 = (1, 1, 2, 2, 3, 5, 5, 5), \qquad D_2 = (1, 1, 3, 3, 4, 4, 7, 7).$$

1. Show that D_1 is graphical (there exists a simple graph whose degree sequence is D_1) and construct explicitly a simple graph G_1 whose degree sequence is D_1.
2. (a) Prove that if there exists a *simple bipartite graph* G whith degree sequence is D_1, then G is a subgraph of the complete bipartite graph $K_{5,3}$, $K_{6,2}$ or $K_{7,1}$.
   (b) Determine whether there exists a *simple bipartite graph* G whose degree sequence is D_1.
3. (a) Determine whether there exists a *graph* G_2 whose degree sequence is D_2, and if so, give an example.
   (b) Prove that D_2 is not graphical; that is, show that there is no simple graph whose degree sequence is D_2.

Question 2

1. Let T be a tree with $n \geq 2$ vertices.
   (a) Show that T has at least two leaves.
   (b) Prove that removing a leaf v from T yields another tree.
2. Let T be a tree with the degree sequence $(x, y, z, 1, 1)$, where $x \geq y \geq z$.
   (a) Prove that $x + y + z = 6$ and solve this equation.
   (b) List all non-isomorphic trees with exactly 5 vertices and exactly 2 leaves. Draw each of them.
3. Find all possible degree sequences for a tree with $n = 5$ vertices.
4. Draw all trees, up to isomorphism, with $n = 5$ vertices.

Question 3

1. (a) Find all integers $p \geq 1$ such that K_p is Eulerian.
   (b) Determine all integers $p \geq 1$ and $q \geq 1$ such that $K_{p,q}$ is Eulerian.
   (c) Find all integers $n \geq 3$ such that the complement $\overline{C_n}$ of the cycle C_n is Eulerian.
2. (a) Find all integers $p \geq 1$ such that K_p is Hamiltonian.
   (b) Determine all integers $p \geq 1$ and $q \geq 1$ such that $K_{p,q}$ is Hamiltonian.
3. Prove that the complement $\overline{C_n}$ of the cycle C_n with $n \geq 3$ vertices, is:
   (a) Not Hamiltonian for $n \in \{3, 4\}$.
   (b) Hamiltonian for $n = 5$.
   (c) Hamiltonian for all $n \geq 6$.
4. Prove that the complement $\overline{C_n}$ of the cycle C_n is Hamiltonian if and only if $n \geq 5$.

Solutions

Solution to Question 1

Let

$$D_1 = (1, 1, 2, 2, 3, 5, 5, 5), \qquad D_2 = (1, 1, 3, 3, 4, 4, 7, 7).$$

be two sequences of integers.

1. **Solution.**

   Showing that D_1 is graphical

   Arrange the sequence in non-increasing order:

   $$(5, 5, 5, 3, 2, 2, 1, 1).$$

   Applying the Havel–Hakimi reduction:

   $$(5, 5, 5, 3, 2, 2, 1, 1) \to (4, 4, 2, 1, 1, 1, 1) \to (3, 1, 1, 1, 0, 0) \to (0, 0, 0, 0, 0).$$

   Since the process ends with the zero sequence, the sequence D_1 is graphical. □

   Solution. Construction of a graph

   Explicit realization of the degree sequence D_1

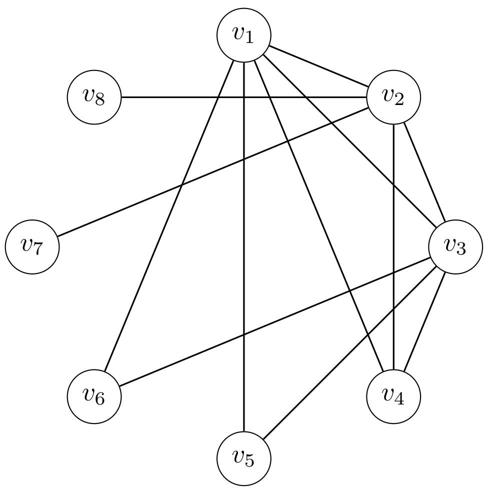
   One realization of the sequence

   $$D_1 = (1, 1, 2, 2, 3, 5, 5, 5)$$

   is obtained with vertices $v_1, \ldots, v_8$ and the following edges:

   $$v_1 \sim v_2, v_3, v_4, v_5, v_6,$$
   $$v_2 \sim v_3, v_4, v_7, v_8,$$
   $$v_3 \sim v_4, v_5, v_6.$$

   The corresponding graph is shown below.



   It is straightforward to verify that the degrees of the vertices on G_1 are

   $$(5, 5, 5, 3, 2, 2, 1, 1),$$

   which corresponds exactly to the sequence D_1. □

2. **Solution. Bipartite case**

   Let $D_1 = (1, 1, 2, 2, 3, 5, 5, 5)$, and suppose that there exists a *simple bipartite graph* $G = (V = A \cup B, E)$ whose degree sequence is D_1. Then

   $$|V| = |A| + |B| = 8,$$

   where A and B form a bipartition of V. Without loss of generality, assume that $|A| \leq |B|$.

   (a) In a bipartite graph, the degree of any vertex is bounded above by the cardinality of the opposite part. Since the maximum degree in D_1 is 5, it follows that

   $$\max\{|A|, |B|\} \geq 5.$$

   Together with the constraint $|A| + |B| = 8$ and the assumption $|A| \leq |B|$, we obtain

   $$3 \leq |A| \leq |B|, \qquad |B| \geq 5.$$

   Consequently, any such graph G must be a subgraph of one of the following complete bipartite graphs:

   $$K_{5,3}, \qquad K_{6,2}, \qquad K_{7,1}.$$

   (b) The degree sequence D_1 contains exactly three vertices of degree 5. In a bipartite graph, a vertex achieves degree 5 if and only if it is adjacent to all vertices in the opposite part. Therefore, the part opposite to these vertices must contain exactly 5 vertices.

   It follows that all vertices of degree 5 belong to the same part, say A, and hence

   $$|B| = 5 \quad \text{and} \quad |A| = 3.$$

   In particular, each vertex in A must be adjacent to every vertex in B, so that G is necessarily a spanning subgraph of $K_{5,3}$ in which all vertices of A have full degree 5.

   This implies that every vertex in B has degree at least 3, since it is adjacent to all three vertices of A. However, this contradicts the fact that the degree sequence D_1 contains vertices of degree 1.

   Therefore, no simple bipartite graph with degree sequence D_1 exists.

   □

3. **Solution. Sequence D_2**

   (a) **Graph G_2 whose degree sequence is D_2**

   Existence of a graph realizing D_2

   Consider the sequence

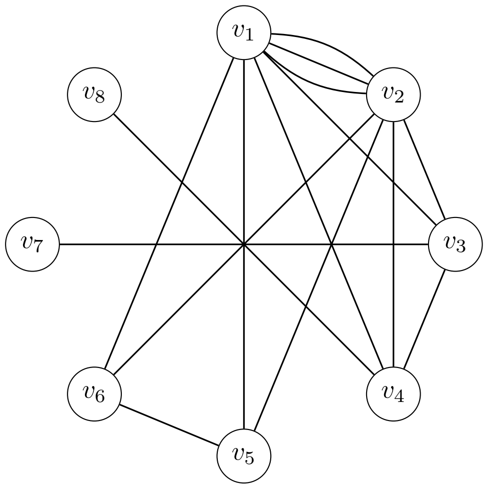
   $$D_2 = (1, 1, 3, 3, 4, 4, 7, 7).$$

   The sum of the degrees is

   $$1 + 1 + 3 + 3 + 4 + 4 + 7 + 7 = 30,$$

   which is even. Hence, by the Handshake Lemma, this sequence is admissible for a graph G_2 (not necessarily simple) whose degree sequence is D_2.

   A multigraph G_2 realizing D_2



Conclusion: There exists a graph G_2 (not necessarily simple) whose degree sequence is D_2.

(b) D_2 **is not graphical**

First method
In a simple graph with 8 vertices the maximum possible degree is 7. If two vertices have degree 7, they must be adjacent to every other vertex.

This forces all remaining vertices to have degree at least 2, which contradicts the presence of vertices of degree 1.

Therefore the sequence D_2 is not graphical.

Second method
Arrange the sequence D_2 in non-increasing order:

$$(7, 7, 4, 4, 3, 3, 1, 1).$$

Applying the Havel–Hakimi reduction:

$$(7, 7, 4, 4, 3, 3, 1, 1) \to (6, 3, 3, 2, 2, 0, 0) \to (2, 2, 1, 1, -1, -1).$$

Since the process ultimately results in the existence of negative integer, the sequence D_2 cannot be graphical.

□

Solution to Question 2

1. Let $T = (V, E)$ be a tree with $n \geq 2$ vertices.

   (a) **Leaves of a tree**
   By contradiction; suppose that, there is $a \in V$, such that $deg(a) \in \{1, 2\}$ and $\forall v \in V \setminus \{a\},\ deg(v) \geq 2$, then the sum of degrees would be at least $2n - 1$, and $2|E| \geq (2n - 1)$, since $2|E|$ is even, hence $2|E| \geq 2n$ implying

   $$|E| \geq n.$$

   However a tree has exactly $n - 1$ edges, giving a contradiction. Hence a tree must contain at least two vertices of degree 1, called leaves.

   (b) **Removing a leaf**
   Let v be a leaf of a tree $T = (V, E)$ and let u be its unique neighbor.
   The graph $T \setminus \{v\} = (V \setminus \{v\}, E \setminus \{u, v\})$ by removing v on a tree T, implies $T \setminus \{v\}$ is a connected graph and without cycle, which is again a tree.

2. **Solution. Trees with $n = 5$ vertices**
Let T be a tree with the degree sequence

$$(x, y, z, 1, 1), \qquad x \geq y \geq z.$$

(T is a tree with 5 vertices.)

(a) **Tree with degree sequence $(x, y, z, 1, 1)$**

Since T is a tree with $n = 5$ vertices, it has exactly $n - 1 = 4$ edges. By the Handshake Lemma,

$$\sum \deg(v) = 2|E| = 8.$$

Hence,

$$x + y + z + 1 + 1 = 8 \quad \Longrightarrow \quad x + y + z = 6 \quad (1).$$

Determine all integer solutions of the equation (1) satisfying

$$x \geq y \geq z \geq 1.$$

The only possibilities are:

$$(4, 1, 1), \quad (3, 2, 1), \quad (2, 2, 2).$$

(b) **Tree with 5 vertices and exactly two leaves**

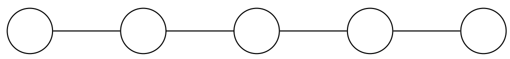
Since the sequence contains exactly two vertices of degree 1, the only valid solution is

$$(x, y, z) = (2, 2, 2).$$

Thus the degree sequence is

$$(2, 2, 2, 1, 1).$$

Conclusion: There is exactly one tree with 5 vertices and exactly two leaves.
Therefore, a tree with 5 vertices and exactly two leaves must be a path; and the path P_5 is the only tree of this type.



□

3. **Solution. All degree sequences for trees with $n = 5$**

We seek all sequences of positive integers summing to 8.

The possible non-increasing sequences are:

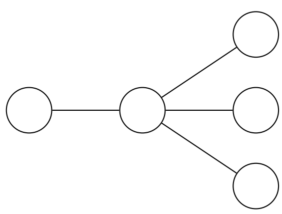
$$(4, 1, 1, 1, 1), \quad (3, 2, 1, 1, 1), \quad (2, 2, 2, 1, 1).$$

All of these sequences are graphical and can be represented as trees. □

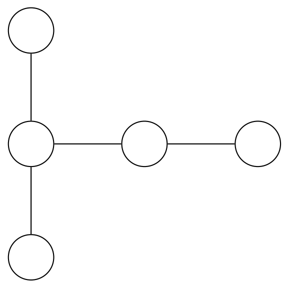
4. **Solution. All trees with 5 vertices (up to isomorphism)**

There are exactly three non-isomorphic trees with 5 vertices:

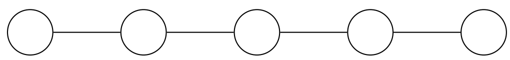
(i) Star $K_{1,4}$ (degree sequence $(4, 1, 1, 1, 1)$)



(ii) The "T-shaped" tree (degree sequence $(3, 2, 1, 1, 1)$)



(iii) Path P_5 (degree sequence $(2, 2, 2, 1, 1)$)



Conclusion: Up to isomorphism, there are exactly three trees with 5 vertices. □

Solution to Question 3

1. **Solution. Eulerian graphs**

   (a) **Complete graphs**
   In the complete graph K_p, each vertex has degree $p - 1$. Recall that a graph is Eulerian if and only if every vertex has even degree and the graph is connected.
   For $p \in \{1, 2\}$, the graph K_p contains no Eulerian circuit.
   For $p \geq 3$, the graph is connected, and thus K_p is Eulerian if and only if

   $$p - 1 \text{ is even} \quad \Longleftrightarrow \quad p \text{ is odd}.$$

   Therefore, K_p is Eulerian if and only if $p \geq 3$ is odd.

   (b) **Complete bipartite graphs**
   Let $K_{p,q} = (X \cup Y, E)$ be a complete bipartite graph. Then

   $$\deg(v) = q \quad (\forall v \in X), \qquad \deg(v) = p \quad (\forall v \in Y).$$

   The graph $K_{p,q}$ is connected if and only if $p \geq 1$ and $q \geq 1$.
   It is Eulerian if and only if all vertex degrees are even, that is,

   $$p \text{ and } q \text{ are even}.$$

   In particular, this excludes the cases $p = 1$ or $q = 1$.
   Therefore, $K_{p,q}$ is Eulerian if and only if $p \geq 2$ and $q \geq 2$ are both even.

   (c) **Complement of a cycle**
   For $n \geq 3$, the complement $\overline{C_n}$ of the cycle C_n is an $(n - 3)$-regular graph.
   For $n \in \{3, 4\}$, the graph $\overline{C_n}$ is disconnected, hence it is not Eulerian.
   For $n \geq 5$, the graph $\overline{C_n}$ is connected, and thus it is Eulerian if and only if

   $$n - 3 \text{ is even} \quad \Longleftrightarrow \quad n \text{ is odd}.$$

   Therefore, $\overline{C_n}$ is Eulerian if and only if $n \geq 5$ is odd.

□

2. **Solution. Hamiltonian graphs**

   (a) **Complete graphs**
   Let $K_p = (V, E)$ be the complete graph on p vertices. Consider $V = \{v_1, ..., v_p\}$.

   If $p \in \{1, 2\}$, then K_p contains no cycle and is therefore not Hamiltonian.
   If $p \geq 3$, the cycle
   $$(v_1, v_2, \ldots, v_p, v_1)$$
   visits every vertex exactly once, hence is a Hamiltonian cycle.
   Therefore, K_p is Hamiltonian if and only if $p \geq 3$.

   (b) **Complete bipartite graphs**
   Let $K_{p,q} = (X \cup Y, E)$ with $|X| = p$ and $|Y| = q$.
   A Hamiltonian cycle in a bipartite graph must alternate between the two parts X and Y.
   Hence, it must contain the same number of vertices in X and in Y, which implies $p = q$.
   If $p = q \geq 2$, a Hamiltonian cycle is given by
   $$(x_1, y_1, x_2, y_2, \ldots, x_p, y_p, x_1).$$
   If $p \neq q$, such a Hamiltonian cycle cannot exist. Moreover, $K_{1,1}$ has no cycle.
   Therefore, $K_{p,q}$ is Hamiltonian if and only if $p = q \geq 2$.

□

3. **Solution. Hamiltonian properties of $\overline{C_n}$**

   Let $n \geq 3$.

   (a) For $n \in \{3, 4\}$, the graph $\overline{C_n}$ is disconnected. Hence, it is not Hamiltonian.

   (b) For $n = 5$, the graph $\overline{C_5}$ is isomorphic to C_5, and is therefore Hamiltonian.

   (c) For $n \geq 6$, the graph $\overline{C_n}$ has minimum degree
   $$\delta(\overline{C_n}) = n - 3.$$
   Since
   $$n - 3 \geq \frac{n}{2},$$
   Dirac's theorem implies that $\overline{C_n}$ is Hamiltonian.

□

4. **Solution.**

   From the previous results, we conclude that
   $$\overline{C_n} \text{ is Hamiltonian if and only if } n \geq 5.$$

□