King Saud University, College of Science
Department of Mathematics
First Semester (2025-2026 G)-(1447- 1448 H)

Second Midterm Exam of Math 5301.
Time allowed: 2 Hours

Question 1.

1. (a) Let $G = (V, E)$ be a graph, and let x be a leaf. Prove that:

$$G \text{ is connected } \iff G - x \text{ is connected.}$$

   (b) Let $T = (V, E)$ be a tree, and let $x \in V$. Show that:
      i. The graph $T - x$ is a forest.
      ii. $T - x$ is a tree if and only if x is a leaf in T.

2. Let T be a tree of order $n \geq 2$.
   (a) Show that T has at least two leaves.
   (b) Assume that T is a tree on $n = 12$ vertices having five leaves and three vertices of degree 3. Find the degrees of the remaining vertices.

3. Let $T = (V, E)$ be a tree of order $n \geq 3$, and let

$$x_i = |\{v \in V : d_T(v) = i\}|.$$

   (a) Prove that:

$$\sum_{i=3}^{n-1} (i-2)x_i = x_1 - 2.$$

   (b) Find all non-isomorphic trees that have x_1 leaves, where $x_1 \leq 4$.
   (c) Find all possible values of x_i, $1 \leq i$, for trees of order n having 5 leaves and 3 vertices of degree 3.

Question 2.

1. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
   (a) Prove that if G is Hamiltonian, then $|V_1| = |V_2|$.
   (b) Prove that if $|V_1| = |V_2| = n \geq 2$ and $\delta(G) > \frac{n}{2}$, then G is Hamiltonian.

2. Determine all integers $p \geq 1$ for which:
   (a) K_p is Eulerian.
   (b) K_p is Hamiltonian.

3. Determine all integers $p, q \geq 1$ for which:
   (a) $K_{p,q}$ is Eulerian.
   (b) $K_{p,q}$ is Hamiltonian.

4. Prove that the complement $\overline{C_n}$ of the cycle C_n is Hamiltonian for all $n \geq 5$.

Solutions

Question 1.

1. (a) Let x be a leaf in G, so $d_G(x) = 1$ and x is adjacent to exactly one vertex y.

   ($\Rightarrow$) If G is connected, then for every pair of vertices $u, v \in V \setminus \{x\}$ there exists a path between u and v in G. Since x is a leaf, all such paths do not need x, hence they remain valid in $G - x$. Therefore, $G - x$ is connected.

   ($\Leftarrow$) Conversely, suppose that $G - x$ is connected. As x is adjacent to y, every vertex in $G - x$ is connected to y, and hence also to x via the edge xy. Thus G is connected. $\square$

   (b) i. Since T is a tree, it is acyclic (without cycle). Removing a vertex cannot create cycles; therefore, $T - x$ is also acyclic. Each connected component of $T - x$ is a tree, so $T - x$ is a forest. $\square$

   ii. If x is a leaf, then removing it and its incident edge leaves a connected, acyclic graph, which is a tree. Conversely, if $T - x$ is a tree, then removing x did not disconnect T, which is possible only if x was incident to exactly one edge — that is, x is a leaf. $\square$

2. (a) Every tree T has $|E| = n - 1$ edges. Consider a longest path in T, say $P = v_1 v_2 \dots v_k$. The endpoints v_1 and v_k cannot have degree greater than 1, otherwise the path could be extended, contradicting maximality. Hence both endpoints are leaves, and T has at least two leaves. $\square$

   (b) Let x_i denote the number of vertices of degree i. Then the sum of degrees in a tree satisfies:

$$\sum_{i=1}^{n-1} i x_i = 2(n-1).$$

   We are given that $x_1 = 5$ (the number of leaves) and $x_3 = 3$, with $n = 12$. Let x denote the number of remaining vertices.

$$x_1 + x + x_3 = 12 \Rightarrow 5 + x + 3 = 12 \Rightarrow x = 4.$$

   Hence, there are 4 vertices whose degrees are neither 1 nor 3.
   Now, let us check the degree sum. For a tree of order 12, we know that:

$$\sum_{i=1}^{n} i x_i = 2(n-1) = 2(12-1) = 22.$$

   Substituting the known values gives:

$$1(5) + 2x_2 + 3(3) + \sum_{i=4}^{11} i x_i = 22.$$

   Simplifying:

$$5 + 2x_2 + 9 + \sum_{i=4}^{11} i x_i = 22,$$

   so

$$2x_2 + \sum_{i=4}^{11} i x_i = 8.$$

   Since no vertex has degree ≥ 5, we set $x_i = 0$ for all $i \geq 5$. Thus:

$$2x_2 + 4x_4 = 8.$$

   Also, since $x_2 + x_4 = 4$ (the 4 remaining vertices), substituting $x_4 = 4 - x_2$ yields:

$$2x_2 + 4(4 - x_2) = 8.$$

Simplifying:

$$2x_2 + 16 - 4x_2 = 8 \Rightarrow x_2 = 4.$$

Hence, $x_4 = 0$.

$$\boxed{\text{Therefore, the remaining four vertices each have degree 2.}}$$

□

3. **(a) Proof of the identity**

In any tree,

$$\sum_{i\geq 1} x_i = n, \qquad \sum_{i\geq 1} i\,x_i = 2(n-1).$$

Subtracting twice the first equation from the second gives:

$$\sum_{i\geq 1}(i-2)x_i = -2.$$

Since the terms for $i = 1, 2$ are negative or zero:

$$(1-2)x_1 + (2-2)x_2 + \sum_{i\geq 3}(i-2)x_i = -2,$$

that is,

$$-x_1 + \sum_{i\geq 3}(i-2)x_i = -2,$$

hence

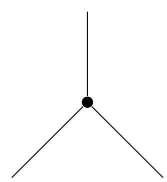
$$\boxed{\sum_{i=3}^{n-1}(i-2)x_i = x_1 - 2.}$$

(b) Classification of trees with few leaves

$\mathbf{x_1 = 2}$: Then $\sum_{i\geq 3}(i-2)x_i = 0$, implying that all vertices have degree 1 or 2. Hence T is a **path** P_n.

Unique type: P_n.

$\mathbf{x_1 = 3}$: Then $\sum_{i\geq 3}(i-2)x_i = 1$. Thus $x_3 = 1$, and all other vertices have degree ≤ 2. The tree has exactly one vertex of degree 3: a **3-spider** (or Y-shaped tree) with three branches of possibly different lengths.



Non-isomorphic trees correspond to unordered triples (l_1, l_2, l_3) of positive integers satisfying $l_1 + l_2 + l_3 = n - 1$.

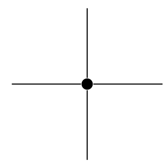
$\mathbf{x_1 = 4}$: Then $\sum_{i\geq 3}(i-2)x_i = 2$. Possible degree combinations:

(a) One vertex of degree 4 (contribution $4 - 2 = 2$): a **4-spider** with four branches.

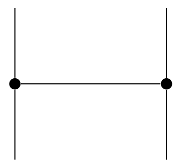
(b) Two vertices of degree 3 (each contributes 1): two connected degree-3 vertices with 4 leaves in total.

Examples:

(1) 4-spider:



(2) Two vertices of degree 3:



(c) Trees with 5 leaves and 3 vertices of degree 3

We are told that:

$$x_1 = 5, \qquad x_3 = 3.$$

By the identity in (a):

$$\sum_{i \geq 3} (i-2)x_i = x_1 - 2 = 3.$$

The vertices of degree 3 contribute exactly $3 \times (3-2) = 3$, so no vertices of higher degree occur:

$$x_i = 0 \quad \text{for all } i \geq 4.$$

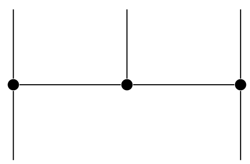
Hence all degrees are 1, 2, or 3. Using the equation $\sum_i x_i = n$:

$$x_2 = n - x_1 - x_3 = n - 8.$$

We must have $x_2 \geq 0 \Rightarrow n \geq 8$.

$$\boxed{x_1 = 5, \quad x_2 = n-8, \quad x_3 = 3, \quad x_i = 0 \ (i \geq 4), \ n \geq 8.}$$

The smallest such tree has $n = 8$ vertices and no degree-2 vertices. Its structure consists of 3 vertices of degree 3 connected in a minimal way so that 5 leaves appear. One possible realization is shown below.



For $n > 8$, inserting degree-2 vertices along any edges preserves x_1 and x_3 while increasing n.

Hence:

$$\boxed{x_1 = 5, \quad x_3 = 3, \quad x_2 = n-8, \quad x_i = 0 \ (i \geq 4), \quad n \geq 8.}$$

Question 2.

1. **Bipartite graphs.**

   (a) A Hamiltonian cycle in a bipartite graph alternates between the two parts V_1 and V_2. Hence, the number of vertices from V_1 equals the number of vertices from V_2 in the cycle. Since a Hamiltonian cycle visits all vertices of G, we must have $|V_1| = |V_2|$. □

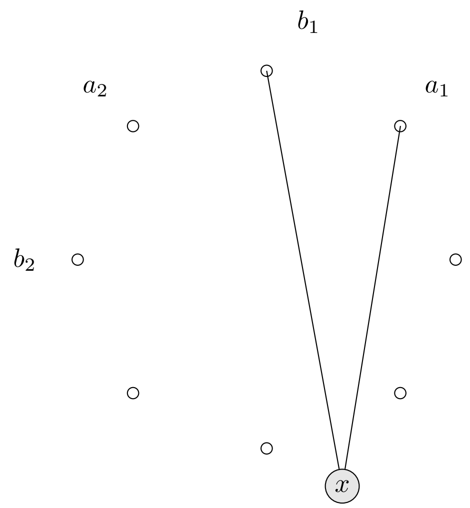
(b) Assume $|V_1| = |V_2| = n$ and $\delta(G) > \frac{n}{2}$, and prove that G is an Hamiltonian graph.

We present a proof using the *maximal cycle* and *insertion* (rotation) technique. Let C be a longest cycle in G and write it as

$$C = a_1 b_1 a_2 b_2 \dots a_t b_t a_1,$$

with $a_i \in V_1$ and $b_i \in V_2$. If $t = n$ then C is Hamiltonian; assume $t \leq n-1$. Pick $x \notin V(C)$; without loss of generality $x \in V_1$. Since $\deg(x) > \frac{n}{2}$ and C contains only t vertices of V_2, x is adjacent to more than $t/2$ vertices among $b_1, \dots, b_t$. Hence two of those neighbors are consecutive on C. That is, there exists i with $xb_i, xb_{i+1} \in E$.

Replace the short segment $b_i a_{i+1} b_{i+1}$ of C by $b_i x b_{i+1}$. This produces a cycle C' that contains x and all vertices of C except possibly a_{i+1}. If a_{i+1} still lies on the new long path then $|C'| > |C|$, contradiction. Otherwise $|C'| = |C|$ but it contains x instead of a_{i+1}. Repeat the procedure; each insertion brings into the cycle a vertex of degree $> \frac{n}{2}$, which again has more than half the neighbors on the other side, ensuring the possibility of further insertions. Since the number of vertices is finite, this process eventually increases the size of the cycle, contradicting maximality of C. Therefore $t = n$ and C is Hamiltonian.



Insertion: replace short segment by x.

2. **Complete graphs K_p.**

(a) In K_p, each vertex has degree $p-1$. A connected graph is Eulerian if and only if every vertex has an even degree. Hence, K_p is Eulerian if $p-1$ is even, i.e., if p is odd. Therefore, K_p is Eulerian for all odd integers $p \geq 1$. □

(b) The complete graph K_p is Hamiltonian for all $p \geq 3$, since it contains a cycle of length p. For $p = 1$ or $p = 2$, a Hamiltonian cycle cannot exist. Hence, K_p is Hamiltonian if and only if $p \geq 3$. □

3. **Complete bipartite graphs $K_{p,q}$.**

(a) In $K_{p,q}$, each vertex in the part of size p has degree q, and each vertex in the part of size q has degree p. The graph is connected for $p, q \geq 1$. It is Eulerian if and only if all vertices have even degree, i.e., when both p and q are even. Thus, $K_{p,q}$ is Eulerian exactly when p and q are even. □

(b) A Hamiltonian cycle in a bipartite graph must alternate between parts. Therefore, the two parts must have the same number of vertices. Thus, $K_{p,q}$ can be Hamiltonian only if $p = q$. Moreover, since a cycle must have at least four vertices, we need $p = q \geq 2$. Hence, $K_{p,q}$ is Hamiltonian if and only if $p = q \geq 2$. □

4. **Complement of a cycle C_n.**

For $n = 5$, the cycle C_5 is self-complementary, so $\overline{C_5} \cong C_5$ is Hamiltonian.

For $n \geq 6$, each vertex of C_n is adjacent to two others, so in the complement $\overline{C_n}$, each vertex has degree $n-3$. Hence,

$$\delta(\overline{C_n}) = n - 3.$$

For $n \geq 6$, we have $n - 3 \geq \frac{n}{2}$. By **Dirac's Theorem**, any graph with n vertices and minimum degree $\delta(G) \geq \frac{n}{2}$ is Hamiltonian. Therefore, $\overline{C_n}$ is Hamiltonian for all $n \geq 6$.

Combining both cases, $\overline{C_n}$ is Hamiltonian for all $n \geq 5$. □