King Saud University, College of Science
Department of Mathematics
Midterm Exam 1447 H - (2025 − 2026 G)

First Midterm Exam of Math 431.
Allotted time: one and a half hours

Let $G = (V, E)$ be a simple finite graph of order $n \geq 2$. Fix $x \in V$. We denote the complement of G by $\overline{G} = (V, \overline{E})$, and the degree of x in G by $d_G(x)$.

Question 1

1. Prove that if G is self-complementary, then its order n satisfies $n = 4p$ or $n = 4p + 1$, for some $p \in \mathbb{N}$.
2. Determine all values of n for which the path P_n (respectively the cycle C_n) is self-complementary.
3. Prove that if there exists $x \in V$ such that $d_G(x) = 0$, then G is not self-complementary.

Question 2

1. (a) Prove that if $d_G(x) \geq 2$ for every $x \in V$, then G contains a cycle.
   (b) Is the converse true? That is, if G contains a cycle, must every vertex satisfy $d_G(x) \geq 2$? Justify your answer.
2. Let G be a connected graph of order n. Prove that if $d_G(x) \leq 2$ for every $x \in V$, then G is isomorphic to either a path P_n or a cycle C_n.

Question 3

Consider the graph

$$G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}).$$

1. (a) Find all walks from v_2 to v_4 of length 4.
   (b) Find all paths from v_2 to v_4 of length 4.
   (c) For $n \geq 5$, is there a path from v_2 to v_4 of length n? Justify your answer.
2. Using two different algorithms on the complete graph K_6, find two spanning trees T_1 and T_2.
3. Prove that T_1 and T_2 are not isomorphic.

Question 4

Consider the two sequences:

$$D_1 = (1, 1, 2, 2, 3, 5, 6, 6), \qquad D_2 = (1, 1, 3, 3, 4, 4, 6, 6).$$

1. Show that D_1 is not graphical (i.e., it is not the degree sequence of any simple graph).
2. (a) Does there exist a simple bipartite graph G_1 such that $\mathrm{DEG}(G_1) = D_2$?
   (b) Show that D_2 is graphical and construct a graph G_2 with degree sequence D_2.