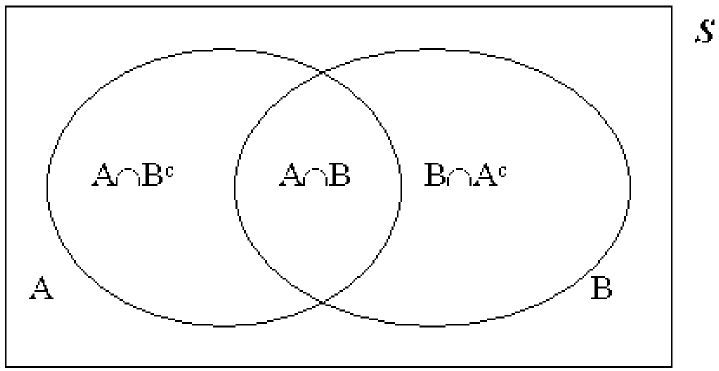
Basic Rules of Probability and Combinatorial Principles

Basic Set Operations

- $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$; $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- A' or A^c = complement of A $= \{x \mid x \notin A\}$; $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$
- $A \subset B = B$ contains all the sample points in event A. It calls A is a subevent of B.
- If $A \subset B$, then $A \cap B = A$ and $A \cup B = B$

Basic Probability

- For a sample space S, $\Pr[S] = 1$; $\Pr[\varnothing] = 0$ where $\varnothing$ is the null or empty set
- For an event A in the sample space S, $1 \geq \Pr[A] \geq 0$
- For any event A, $\Pr[A] = 1 - \Pr[A^c]$
- If A and B are two mutually exclusive events, (no sample points in common $A \cap B = \varnothing$), then $\Pr[A \cap B] = 0$
- **General Formula :** $\boxed{\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]}$
- $\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C] + \Pr[A \cap B \cap C]$
- If $A \subset B$ then $\Pr[A] \leq \Pr[B]$
- $\Pr[A] = \Pr[A \cap B] + \Pr[A \cap B^c]$



Conditional Probability

- For two events A and B, the conditional probability of A given B has occurred is:

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

- If $A \subset B$, then $\Pr[A \mid B] = \dfrac{\Pr[A]}{\Pr[B]}$ and $\Pr[B \mid A] = 1$
- $\Pr[A^c \mid B] = 1 - \Pr[A \mid B]$ and If $A_1 \cap A_2 = \varnothing$ then $\Pr[A_1 \cup A_2 \mid B] = \Pr[A_1 \mid B] + \Pr[A_1 \mid B]$

  i.e. $\Pr[\,\cdot \mid B]$ is a probability on S.

Baye's Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of n mutually exclusive and exhaustive events with $\Pr[A_i] > 0$ for $i = 1, \ldots, n$.
Then, for any other event B,

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \mid A_i] \; \Pr[A_i]$$

$$\Pr[A_k \mid B] = \frac{\Pr[A_k \cap B]}{\Pr[B]} = \frac{\Pr[B \mid A_k] \; \Pr[A_k]}{\sum_{i=1}^{n} Pr[B \mid A_i] \; \Pr[A_i]} \quad k = 1, \ldots, n$$

Independence

Two events A and B are independent if and only if $\Pr[B \mid A] = \Pr[B]$

if and only if $\Pr[A \mid B] = \Pr[A]$

if and only if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

Combinatorial Principles

Fundamental Principle of Counting: (also known as the multiplication rule for counting) If a task can be performed in n1 ways, and for each of these a second task can be performed in n2 ways, and for each of the latter a third task can be performed in n3 ways, ... , and for each of the latter a kth task can be performed in nk ways, then the entire sequence of k tasks can be performed in

n1 • n2 • n3 • ... • nk ways.

Permutations

An ordered sequence of k objects taken from a set of n distinct objects is called a permutation of the objects of size k. P_k^n denotes the number of size k permutations that can be constructed from the n objects.

$$\mathrm{P}_k^n = \frac{n!}{(n-k)!}$$

Combinations

An unordered subset of k objects taken from a set of n distinct objects is called a combination. $\binom{n}{k}$ denotes the number of combinations of size k that can be formed from n objects.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{0} = 1 \qquad \binom{n}{1} = \binom{n}{n-1} = n \qquad \binom{n}{k} = \binom{n}{n-k}$$

Random Variables and Probability Distributions, Expectation and Other Distributional Parameters

Random Variables Probability Distributions and Independence

- Given an experiment with sample space S, a random variable X is any rule that associates a number with each outcome in S.
- The cumulative distribution function (cdf) of a random variable written $F(x)$ is defined for every number x by: $F(x) = \Pr[X \leq x]$
- Two random variables X and Y are independent if

  $\Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B]$ for every A and B.

Discrete Random Variables

- A set is described as discrete if its elements can be listed in sequence or if it consists of a finite number of elements.
- The probability distribution or probability mass function of a discrete random variable written $p(x)$ is defined for every number by: $p(x) = \Pr[X = x]$
- The probability mass function $p(x)$ must satisfy: $0 \leq p(x) \leq 1$ and $\sum_x p(x) = 1.$
- The cumulative distribution function (cdf) of a discrete random variable with pmf $p(x)$ written $F(x)$ is defined by:

$$F(x) = \sum_{k \leq x} \Pr[X = k] \quad \text{for every number x}$$

- Two random discrete variables X and Y are independent if $\Pr[X=x, Y=y] = \Pr[X=x] \Pr[Y=y]$ for every x and y.

Continuous Random Variables

- A random variable X is continuous if its set of possible values is an entire interval of numbers.
- The probability distribution or probability density function (pdf) of a continuous random variable written $f(x)$ is :

$$\Pr[a \leq X \leq b] = \int_a^b f(x)\, dx$$

- The probability density function $f(x)$ must satisfy: $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1.$
- The cumulative distribution function of a continuous random variable written $F(x)$ is: $F(x) = \Pr[X \leq x] = \int_{-\infty}^{x} f(y)\, dy$
- We have $\Pr[a \leq x \leq b] = F(b) - F(a)$ and $F'(x) = f(x)$

Expectation, Variance, Standard deviation, Covariance

- The expected value or mean value of a discrete random variable written $E(X)$ or μ_X is:

$$E(X) = \sum_x x \, \Pr[X = x] \text{ if } X \text{ is dicrete and } \quad E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx \quad \text{if } X \text{ is continuous.}$$

- The variance of a discrete random variable written $Var(X)$ or σ_X^2 is: $Var(X) = E[(X - E(X))^2] = E[X^2] - [E(X)]^2$

$$Var(X) = \sum_x (x - \mu_X)^2 \, \Pr[X = x] \text{ if } X \text{ is dicrete and } \quad Var(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) \, dx \quad \text{if } X \text{ is continuous.}$$

- $\sigma_X = \text{standard deviation} = \sqrt{Var(X)}$
- $Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y) = Cov(Y,X)$

If X, Y are independent, then $Cov(X,Y) = 0$

Rules of expected value:

- For any constant a, $E(aX) = aE(X)$
- For any random variables X, Y, $E(X + Y) = E(X) + E(Y)$

Rules of variance:

- For any constants a and b, $Var(aX + b) = a^2[Var(X)]$
- For any random variables X, Y, $Var(X + Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

If X, Y are independent, then $Var(X + Y) = Var(X) + Var(Y)$

Expected value of a function of a random variable

$E[h(X)] = \sum_x h(x)\, \Pr[X = x]$ if X is dicrete and $E[h(X)] = \int_{-\infty}^{\infty} h(x)\, f(x)\, dx$ if X is continuous.

Moment generating function

The moment generating function of a discrete random variable written $M_X(t)$ is: $M_X(t) = E(e^{tx})$

- **Expectation, variance, and moments via mgf's:** $M_X(0) = 1$,

  $M_X{}'(0) = \mathrm{E}(X)$, $M_X{}''(0) = \mathrm{E}(X^2)$, $M_X{}'''(0) = \mathrm{E}(X^3)$, etc. $\mathrm{Var}(X) = M_X{}''(0) - M_X{}'(0)^2$.

- **Mgf of a multiple of a r.v.:** If X has mgf $M_X(t)$, and $Y = cX$ with c a constant, then the mgf of Y is

$$M_Y(t) = E(e^{tY}) = E(e^{tcX}) = M_X(tc).$$

- **Mgf of a sum of independent r.v.'s X and Y:** If X and Y are independent, then $X + Y$ has mgf

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

  (An analogous formula holds for sums of more than two independent r.v.'s.)

Other parameters of the distribution

- For $0 \le p \le 1$, the $(100p)^{th}$ percentile of a continuous random variable written $\eta(p)$ is:

$$p = F[\eta(p)] = \int_{-\infty}^{\eta(p)} f(y)\ y$$

- The median of a continuous distribution written $\tilde{\mu}$ satisfies $0.5 = F(\tilde{\mu}) = \mathit{the}\ 50^{th}$ percentile of the distribution.
- The mode of a distribution is any point m at which the probability mass function $p(x)$ or the density function $f(x)$ is maximized.
- The skewness of a distribution is: $\dfrac{E[(X - \mu)^3]}{\sigma^3}$

Discrete Distributions

1. ***Uniform*** A distribution of N points, $1, 2 \ldots, N$, where N is an integer.

$$p(x) = \frac{1}{N} \qquad E(X) = \frac{N+1}{2} \qquad Var(X) = \frac{N^2-1}{12} \qquad M_X(t) = \frac{e^t(e^{Nt}-1)}{N\,(e^t-1)}$$

2. ***Binomial*** An experiment consisting of a fixed number of n trials. Each trial is identical and results in one of two possible outcomes, denoted success (S) or failure (F) The trials are independent and the probability of a success is denoted p. The binomial random variable X is equal to the number of successes in n trials.

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x};\; x = 0, 1, 2, \ldots, n, \qquad E(X) = np \qquad Var(X) = np(1-p) \qquad M_X(t) = (1 - p + p\,e^t)^n$$

3. ***Hypergeometric*** Closely related to binomial, but it is known that the population (N) contains M successes. X is the number of successes in a random sample of size n drawn from a population of M successes.

$$p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{M}} \qquad E(X) = n\,\frac{M}{N} \qquad Var(X) = n\,\frac{M}{N}\,\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$$

4. ***Pascal*** Related to binomial. An experiment continues until a total of r successes have been observed (r is a positive integer). X is equal to the number of trials until the r^{th} success is completed.

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r};\; x = r,\, r+1, r+2, \ldots \qquad E(X) = \frac{r}{p} \qquad Var(X) = \frac{r(1-p)}{p^2} \qquad M_X(t) = \left[\frac{p\,e^t}{1-(1-p)\,e^t}\right]^r$$

5. ***Geometric*** Special case of negative binomial when r = 1. So, want to perform the experiment until a success occurs. X is equal to the number of trials until the first success is completed.

$$p(x) = p(1-p)^{x-1},\; x = 1, 2, \ldots \qquad E(X) = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2} \qquad M_X(t) = \frac{p\,e^t}{1-(1-p)e^t}$$

6. ***Poisson*** Not based on any simple experiment. Most often used to count the number of a certain type of event that occurs in a period of time.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!},\; x = 0, 1, 2, \ldots \qquad E(X) = \lambda \qquad Var(X) = \lambda \qquad M_X(t) = e^{\lambda(e^t-1)}$$

for some $\lambda > 0$ where λ is frequently some rate per unit of time.

Continuous Distributions

1. ***Uniform*** X is distributed uniformly over (a, b).

$$f(x) = \frac{1}{b-a}\ ,\ a<x<b \qquad E(X) = \frac{b-a}{2} \qquad Var(X) = \frac{(b-a)^2}{12} \qquad M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)\ t}$$

2. ***Normal*** Distribution given that is the mean and is the variance of the distribution.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad E(X) = \mu \qquad Var(X) = \sigma^2 \qquad M_X(t) = exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$$

When μ = 0 and σ = 1, the normal distribution is called a standard normal distribution and the standard normal random variable is denoted Z.

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

If X has a normal distribution, it can be standardized by : $Z = \frac{X-\mu}{\sigma}$

3 ***Gamma*** For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$.

$$\Gamma(\alpha+1) = \alpha\ \Gamma(\alpha) \quad \text{and} \quad \Gamma(n) = (n-1)! \quad \text{for } n \text{ positive integer.}$$

A continuous random variable X is said to have a gamma distribution with parameters $\alpha, \beta > 0$ if:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},\ x > 0 \qquad E(X) = \frac{\alpha}{\beta} \qquad Var(X) = \frac{\alpha}{\beta^2} \qquad M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \ for\ \ t < \beta$$

4. ***Exponential*** The exponential distribution is a special case of the gamma distribution with $\alpha = 1$ and $\beta = \lambda$

$$f(x) = \lambda e^{-\lambda x},\ x > 0 \qquad E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad M_X(t) = \frac{\lambda}{\lambda - t}\ for\ \ t < \lambda$$

Joint, Marginal and Conditional Distributions

- Let X and Y be two discrete random variables defined on the sample space S of an experiment. The joint probability mass function $p_{X,Y}(x,y)$, is defined for each (x,y) pair by:

$$p_{X,Y}(x,y) = p(x,y) = \Pr[X = x, \; Y = y]$$

and $p(x,y)$ must satisfy:

$$0 \leq p(x,y) \leq 1 \text{ and } \sum_x \sum_y p(x,y) = 1.$$

- Let X and Y be two continuous random variables. Then, the joint probability density function $f_{X,Y}(x,y) = f(x,y)$ must satisfy:

$$f(x,y) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

Marginal Distribution

- The marginal proability mass function for the discrete random variables X and Y, denoted $p_X(x)$ and $p_Y(y)$ are given by:

$$p_X(x) = \sum_y p(x,y) \quad \text{and} \quad p_Y(y) = \sum_x p(x,y)$$

- In the continuous case, they are given by:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Expected Value

Let X and Y be two jointly distributed random variables. The expected value of a function h of (X, Y) denoted $E[h(X,Y)]$ is:

$$E[h(X,Y)] = \sum_x \sum_y h(x,y)\ p(x,y) \qquad \text{If } X \text{ and } Y \text{ are discrete}$$

$$E[h(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y)\ f(x,y)\ dx\, dy \qquad \text{If } X \text{ and } Y \text{ are continuous}$$

Moment Generating Function

The moment generating function for two jointly distributed random variables X and Y, denoted $M_{X,Y}(t_1, t_2)$ is:

$$M_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}]$$

Variance, Covariance, Correlation

- The covariance between two random variables X and Y is :

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

$Cov(X,Y) = \sum_x \sum_y x\,y\ \ p(x,y) - \mu_X \mu_Y$ in the discrete case

$Cov(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\,y\ f(x,y)\,dx\,dy - \mu_X \mu_Y$ in the continuous case

- If X and Y are independent, then $E[X \cdot Y] = E[X]E[Y]$ $\quad Cov(X,Y) = 0$
- $Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X,Y]$
- The correlation coefficient of X and Y, denoted by $corr(X,Y)$ or $\rho_{X,Y}$ is defined by:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

If a and c are either both positive or both negative, $Corr(aX+b, cY+d) = Corr(X,Y)$

For any two random variables X and Y, $-1 \le Corr(X,Y) \le 1$.

Independence

Two random variables X and Y are said to be independent if and only if one of the following holds:

(i) for every pair of x and y values:

$p(x,y) = p_X(x)\, p_Y(y)$ if X is dicrete and $f(x,y) = f_X(x)\, f_Y(y)$ if X is continuous.

(ii) for every pair of functions g and h

$$E[g(X)\, h(Y)] = E[g(X)]\, E[h(Y)]$$

(iii) for every pair of t_1 and t_2 values:

$$M_{X,Y}(t_1, t_2) = M_X(t_1)\, M_Y(t_2)$$

Conditional Distributions in the continuous case

Let X and Y be two random variables with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$ For any x-value for which $f_X(x) > 0$, the conditional probability density function of Y given that $X = x$ is:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

Conditional expectation and variance

For two jointly distributed random variables X and Y:

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y=y}(x)\, dx = g(y)$$

$$E[X \mid Y] = g(Y) \qquad Var[X \mid Y] = E[X^2 \mid Y] - E[X \mid Y]^2$$

$$E[X] = E[E[X \mid Y]] \qquad Var[X] = E[Var[X \mid Y]] + Var[E[X \mid Y]]$$