King Saud University

College of Computer and Information Sciences (CCIS)

Computer Science Department

Discrete Math for Computer Science

Midterm - CSC 281- Semester 2 (1447)

Time: 1 Hour and 30 minutes **Total Marks: 25**

Instructions:

- Answer **all questions**.
- Justify all answers with clear and rigorous arguments.
- Logical reasoning must be clearly presented.
- **The examination contain 7 pages.**

Answer Sheet	
Student Name:	
Student ID:	
Section No:	
Student Serial No:	

Question No	**CLO**	**Points**	**Student's Points**
Question 1	CLO1	7	
Question 2	CLO3	6	
Question 3	CLO2	6	
Question 4	CLO4	6	
Total		25	

Question 1 (7 marks): Logic and Predicate Calculus

(a) Satisfiability (2 marks)

Determine whether each of the following propositions is **satisfiable**. Justify your answer.

(i) $(p \lor q \lor r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor q \lor \neg r) \land (\neg p \lor \neg q \lor r)$.

Solution.

Consider the formula

$$\varphi = (p \lor q \lor r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor q \lor \neg r) \land (\neg p \lor \neg q \lor r).$$

Each clause is false for exactly one assignment:

$$\begin{aligned}
(p \lor q \lor r) &\text{ is false if and only if } (p,q,r) = (F,F,F),\\
(p \lor \neg q \lor \neg r) &\text{ is false if and only if } (p,q,r) = (F,T,T),\\
(\neg p \lor q \lor \neg r) &\text{ is false if and only if } (p,q,r) = (T,F,T),\\
(\neg p \lor \neg q \lor r) &\text{ is false if and only if } (p,q,r) = (T,T,F).
\end{aligned}$$

Hence, the formula φ is false if and only if at least one of the above clauses is false, that is, for one of the four assignments listed above.

Since there are $2^3 = 8$ possible truth assignments in total, and only four of them falsify φ, the remaining assignments satisfy φ. Therefore, φ is satisfiable.

For instance, for $(p,q,r) = (T,T,T)$, all clauses evaluate to true, and thus φ is true.

□

(ii) $(p \lor q \lor \neg r) \land (p \lor \neg q \lor r) \land (\neg p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r) \land (p \lor q \lor r)$.

Solution.

Consider the formula

$$\psi = (p \lor q \lor \neg r) \land (p \lor \neg q \lor r) \land (\neg p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r) \land (p \lor q \lor r).$$

Each clause is false for exactly one assignment:

$$\begin{aligned}
(p \lor q \lor \neg r) &\text{ is false if and only if } (p,q,r) = (F,F,T),\\
(p \lor \neg q \lor r) &\text{ is false if and only if } (p,q,r) = (F,T,F),\\
(\neg p \lor q \lor r) &\text{ is false if and only if } (p,q,r) = (T,F,F),\\
(\neg p \lor \neg q \lor \neg r) &\text{ is false if and only if } (p,q,r) = (T,T,T),\\
(p \lor q \lor r) &\text{ is false if and only if } (p,q,r) = (F,F,F).
\end{aligned}$$

Thus, the formula ψ is false precisely for the five assignments listed above.

Since there are $2^3 = 8$ possible assignments, it follows that $8 - 5 = 3$ assignments satisfy ψ. Therefore, ψ is satisfiable.

For example, for $(p,q,r) = (T,T,F)$, all clauses are satisfied, hence ψ evaluates to true.

□

(b) Tautology (1.5 marks)

Use a truth table to determine whether:

$$\big((p \to q) \land (\neg q \to r)\big) \to (p \to r)$$

is a tautology.

Solution.

Consider the formula

$$\varphi = \big((p \to q) \land (\neg q \to r)\big) \to (p \to r).$$

To determine whether φ is a tautology, it suffices to find a truth assignment for which it evaluates to false.

Recall that an implication $A \to B$ is false if and only if A is true and B is false.

Required evidence

Consider the formula

$$\varphi = \big((p \to q) \land (\neg q \to r)\big) \to (p \to r).$$

We construct the full truth table.

p	q	r	$p \to q$	$\neg q$	$\neg q \to r$	$(p \to q) \land (\neg q \to r)$	$p \to r$	φ
T	T	T	T	F	T	T	T	T
T	T	F	T	F	T	T	F	F
T	F	T	F	T	T	F	T	T
T	F	F	F	T	F	F	F	T
F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

From the table, we observe that the formula φ is false for the assignment $(p, q, r) = (T, T, F)$ and true for all other assignments.

Therefore, φ is **not a tautology**.

Remark: Second Method Let $(p, q, r) = (T, T, F)$. Then:

$$p \to q = T \to T = T,$$
$$\neg q \to r = F \to F = T.$$

Hence, the antecedent $(p \to q) \land (\neg q \to r)$ is true.

On the other hand,

$$p \to r = T \to F = F.$$

Therefore, the implication φ is false for this assignment. Consequently, φ is **not a tautology**.

□

(c) Quantifiers (1.5 marks)

Let $Q(x, y)$: "$x - y = 0$" be a predicate with domain $\mathbb{Z}^+$. Determine whether each statement is **true or false**, with justification:

(i) $\exists x\, \forall y\; Q(x, y)$
(ii) $\forall x\; Q(x, 1)$
(iii) $\exists y\, \forall x\; Q(x, y)$
(iv) $\forall y\, \exists x\; Q(x, y)$

Solution.

Let $Q(x, y)$ be the predicate defined by $Q(x, y) : x - y = 0$ over the domain $\mathbb{Z}^+$. Equivalently, $Q(x, y)$ holds if and only if $x = y$.

We examine each statement in turn.

(i) $\exists x\, \forall y\, Q(x, y)$.

This statement asserts that there exists a non negative integer x such that $x = y$ for all a non negative integers y. This is impossible, since a fixed integer cannot be equal to all non negative integers. Hence, the statement is **false**.

(ii) $\forall x\, Q(x, 1)$.

This statement asserts that for all non negative integers x, one has $x = 1$. This is clearly false, since there exist a non negative integers distinct from 1. Hence, the statement is **false**.

(iii) $\exists y\, \forall x\, Q(x, y)$.

This statement asserts that there exists a non negative integer y such that $x = y$ for all non negative integers x. This is equivalent to statement (i) and is false for the same reason. Hence, the statement is **false**.

(iv) $\forall y\, \exists x\, Q(x, y)$.

This statement asserts that for every non negative integer y, there exists a non negative integer x such that $x = y$. This is clearly true, since for each y, one can take $x = y$. Hence, the statement is **true**.

□

(d) Predicate Logic Translation (2 marks)

Domain: set of all people.

- $S(x)$: x has been a student of CSC281
- $A(x)$: x obtained grade A in CSC281
- $T(x)$: x is a TA (teaching assistant) of CSC281

Translate into predicate logic:

(i) Every CSC281 TA who was also a student obtained an A.

Solution.

$$\forall x \left((T(x) \wedge S(x)) \rightarrow A(x)\right).$$

□

(ii) There exists a CSC281 TA who did not obtain an A.

Solution.

$$\exists x \left(T(x) \wedge \neg A(x)\right).$$

□

Question 2 (6 marks): Proofs

(a) Parity Equivalence (4 marks)

Let $n \in \mathbb{Z}^+$. Prove that:

$$n \text{ is even} \iff 3n+4 \text{ is even}.$$

Solution.

Let $n \in \mathbb{Z}^+$. We prove the equivalence by establishing both implications.

($\Rightarrow$) Suppose that n is even. Then there exists $k \in \mathbb{Z}^+$ such that $n = 2k$. It follows that

$$3n+4 = 3(2k)+4 = 6k+4 = 2(3k+2), \quad 3k+2 \in \mathbb{Z}^+$$

which is even.

($\Leftarrow$) We prove the converse by contraposition. Suppose that n is odd. Then there exists $k \in \mathbb{Z}^+$ such that $n = 2k+1$. Hence,

$$3n+4 = 3(2k+1)+4 = 6k+3+4 = 6k+7 = 2(3k+3)+1, \quad 3k+3 \in \mathbb{Z}^+$$

which is odd.

Thus, if $3n+4$ is even, then n cannot be odd, and therefore n must be even.

We conclude that

$$n \text{ is even} \iff 3n+4 \text{ is even}.$$

□

(b) Prime Numbers (2 marks)

Prove that there is no largest prime number.

Solution.

We proceed by contradiction. Suppose that there are only finitely many prime numbers, say

$$p_1, p_2, \ldots, p_n,$$

and that these constitute the complete set of all prime integers.

Consider the integer

$$N = p_1 p_2 \cdots p_n + 1.$$

For each $i \in \{1, \ldots, n\}$, we have

$$N \equiv 1 \pmod{p_i},$$

and therefore $p_i \nmid N$. It follows that none of the primes $p_1, \ldots, p_n$ divides N.

However, since $N > 1$, the Fundamental Theorem of Arithmetic ensures that N admits at least one prime divisor. This prime cannot belong to the list $p_1, \ldots, p_n$, which contradicts the assumption that these are all the prime numbers.

Hence, there exist infinitely many prime numbers.

□

Question 3 (6 marks): Sets, Sequences, and Functions

(a) Set Operations (2 marks)

Let:

$$A = \{1, 3, 5, 7, 9\}, \quad B = \{0, 1, 2, 3, 4, 5\}, \quad C = \{5, 6, 7, 8, 9\}.$$

Compute:

(i) $(A \cup B) \cap C$

Solution.

$$A \cup B = \{0, 1, 2, 3, 4, 5, 7, 9\}, \quad (A \cup B) \cap C = \{5, 7, 9\}.$$

□

(ii) $\{2, 4, 6\} \oplus \{1, 2, 3, 4\}$

Solution.

$$\{2, 4, 6\} \oplus \{1, 2, 3, 4\} = \{1, 3, 6\}.$$

□

(b) Sequences and Recurrence (2 marks)

(i) Evaluate:

$$\sum_{k=1}^{5} \left(2k - (-1)^k\right)$$

Solution. We evaluate the sum

$$\sum_{k=1}^{5} \left(2k - (-1)^k\right).$$

Compute each term:

$$\begin{aligned}
k = 1 &: \ 2(1) - (-1)^1 = 2 + 1 = 3, \\
k = 2 &: \ 2(2) - (-1)^2 = 4 - 1 = 3, \\
k = 3 &: \ 2(3) - (-1)^3 = 6 + 1 = 7, \\
k = 4 &: \ 2(4) - (-1)^4 = 8 - 1 = 7, \\
k = 5 &: \ 2(5) - (-1)^5 = 10 + 1 = 11.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{5} \left(2k - (-1)^k\right) = 3 + 3 + 7 + 7 + 11 = 31.$$

□

(ii) List the first five terms of:

$$a_n = n^2 - n, \ n \geq 1$$

Solution. The sequence is defined by $a_n = n^2 - n$ for $n \geq 1$. We compute the first five terms:

$$a_1 = 1^2 - 1 = 0,$$
$$a_2 = 2^2 - 2 = 2,$$
$$a_3 = 3^2 - 3 = 6,$$
$$a_4 = 4^2 - 4 = 12,$$
$$a_5 = 5^2 - 5 = 20.$$

□

(iii) Consider:

$$a_n = 3a_{n-1} + 2, \ n \geq 1, \quad a_0 = 2,$$

- Compute a_1, a_2, a_3, a_4

  Solution. The sequence is defined recursively by

  $$a_n = 3a_{n-1} + 2, \ n \geq 1, \quad a_0 = 2.$$

  We compute:

  $$a_1 = 3a_0 + 2 = 3 \cdot 2 + 2 = 8,$$
  $$a_2 = 3a_1 + 2 = 3 \cdot 8 + 2 = 26,$$
  $$a_3 = 3a_2 + 2 = 3 \cdot 26 + 2 = 80,$$
  $$a_4 = 3a_3 + 2 = 3 \cdot 80 + 2 = 242.$$

  □

- Find a closed-form expression (Hint: $\sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$).

  Solution. We determine a closed-form expression for the recurrence

  $$a_n = 3a_{n-1} + 2, \ n \geq 1, \quad a_0 = 2.$$

  By iterating the recurrence, we obtain

  $$a_n = 3a_{n-1}+2 = 3^2a_{n-2}+2(3^1+3^0) = ... = 3^{n-1}a_1+2\sum_{i=0}^{n-2} 3^i = 3^{n-1}(3.a_0+2)+2\sum_{i=0}^{n-2} 3^i$$

  Hence

  $$a_n = 3^n a_0 + 2\sum_{i=0}^{n-1} 3^i.$$

Using the identity

$$\sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2},$$

it follows that

$$a_n = 3^n \cdot 2 + 2 \cdot \frac{3^n - 1}{2} = 2 \cdot 3^n + (3^n - 1) = 3^{n+1} - 1.$$

Thus, the closed-form expression is

$$a_n = 3^{n+1} - 1, \ \forall n \geq 0.$$

□

(c) Functions (2 marks)

Let $g, h : \mathbb{R} \to \mathbb{R}$ be the functions defined by:

$$g(x) = 2x, \quad h(x) = 3x - 3$$

(i) Prove that g is bijective (one to one correspondence).

Solution. We prove that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x$ is bijective.

Injectivity. Let $x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2)$. Then

$$2x_1 = 2x_2 \implies x_1 = x_2.$$

Hence, g is injective.

Surjectivity. Let $y \in \mathbb{R}$. We seek $x \in \mathbb{R}$ such that $g(x) = y$. Solving

$$2x = y \implies x = \frac{y}{2},$$

and since $\frac{y}{2} \in \mathbb{R}$, it follows that g is surjective.

Therefore, g is bijective.

□

(ii) Find $g \circ h$.

Solution. We compute the composition $g \circ h$.

For any $x \in \mathbb{R}$,

$$(g \circ h)(x) = g\big(h(x)\big) = g(3x - 3) = 2(3x - 3) = 6x - 6.$$

Thus,

$$(g \circ h)(x) = 6x - 6.$$

Therefore $g \circ h : \mathbb{R} \to \mathbb{R}$ is the functions defined by:

$$g \circ h(x) = 6x - 6$$

□

Question 4 (6 marks): Number Theory

(a) Division Algorithm (1 mark)

Find the quotient and remainder when 29 is divided by 6.
Solution.
By the Euclidean division theorem, there exist unique integers q and r such that

$$29 = 6q + r, \quad \text{with } 0 \leq r < 6.$$

A direct computation gives

$$29 = 6 \cdot 4 + 5,$$

where $0 \leq 5 < 6$.
Therefore, the quotient is $q = 4$ and the remainder is $r = 5$.

□

(b) Modular Arithmetic (2 marks)

Compute:

$$20 \bmod 6, \quad -50 \bmod 9$$

Solution.
We compute each remainder using the Euclidean division.

(i) For 20 mod 6:
By the Euclidean division theorem, there exist integers q and r such that

$$20 = 6q + r, \quad 0 \leq r < 6.$$

A direct computation yields

$$20 = 6 \cdot 3 + 2,$$

with $0 \leq 2 < 6$. Hence,

$$20 \bmod 6 = 2.$$

(ii) For -50 mod 9:
We seek integers q and r such that

$$-50 = 9q + r, \quad 0 \leq r < 9.$$

Taking $q = -6$, we obtain

$$-50 = 9(-6) + 4,$$

with $0 \leq 4 < 9$. Hence,

$$-50 \bmod 9 = 4.$$

□

Determine whether:

$$100 \equiv 4 \pmod 8$$

Solution.
We determine whether

$$100 \equiv 4 \pmod 8.$$

By definition, $100 \equiv 4 \pmod 8$ if and only if $8 \mid (100 - 4)$.
We compute

$$100 - 4 = 96.$$

Since $96 = 8 \cdot 12$, it follows that $8 \mid 96$.
Therefore,

$$100 \equiv 4 \pmod 8.$$

□

(c) Congruences (1 mark)

Let $a \equiv 7 \pmod 9$, $b \equiv 5 \pmod 9$. Find $c \in \{0, \ldots, 8\}$ such that:

$$c \equiv ab \pmod 9.$$

Solution.
Given that $a \equiv 7 \pmod 9$ and $b \equiv 5 \pmod 9$, we use the compatibility of congruences with multiplication:

$$ab \equiv 7 \cdot 5 \pmod 9.$$

We compute

$$7 \cdot 5 = 35.$$

By the Euclidean division,

$$35 = 9 \cdot 3 + 8,$$

with $0 \leq 8 < 9$. Hence,

$$35 \equiv 8 \pmod 9.$$

Therefore,

$$ab \equiv 8 \pmod 9.$$

Since $c \in \{0, \ldots, 8\}$, we conclude that

$$c = 8.$$

□

(d) Base Conversion (1 mark)

Determine the values of x and y, such that:

$$(101101)_2 = (x)_{10}, \quad (3F2)_{16} = (y)_2$$

Solution.
We determine the values of x and y by performing base conversions.

(i) Conversion of $(101101)_2$ into base 10:

$$(101101)_2 = 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0.$$
$$= 32 + 0 + 8 + 4 + 0 + 1 = 45.$$

Thus,

$$x = 45.$$

(ii) Conversion of $(3F2)_{16}$ into base 2:
Each hexadecimal digit corresponds to 4 binary digits:

$$3 = 0011, \quad F = 1111, \quad 2 = 0010.$$

Hence,

$$(3F2)_{16} = (0011\ 1111\ 0010)_2.$$

Removing leading zeros, we obtain

$$(3F2)_{16} = (1111110010)_2,$$

and therefore

$$y = 1111110010.$$

□

(e) Modular Exponentiation (1 mark)

Compute:

$$50^{100} \bmod 13$$

using the modular exponentiation algorithm. Show all steps.
Solution.
We compute

$$50^{100} \bmod 13$$

using modular exponentiation.

First, reduce the base modulo 13: $50 \equiv 11 \pmod{13}$.

Hence, $50^{100} \equiv 11^{100} \pmod{13}$.
Next, express the exponent in binary:

$$100 = (1100100)_2 = 2^6 + 2^5 + 2^2.$$

Thus, $11^{100} = 11^{2^6} \cdot 11^{2^5} \cdot 11^{2^2}$.
We now compute successive squares modulo 13:

$$\begin{aligned}
11^2 &\equiv 121 \equiv 4 \pmod{13},\\
11^4 &\equiv 4^2 = 16 \equiv 3 \pmod{13},\\
11^8 &\equiv 3^2 = 9 \pmod{13},\\
11^{16} &\equiv 9^2 = 81 \equiv 3 \pmod{13},\\
11^{32} &\equiv 3^2 = 9 \pmod{13},\\
11^{64} &\equiv 9^2 = 81 \equiv 3 \pmod{13}.
\end{aligned}$$

Therefore,

$$11^{100} = 11^{64} \cdot 11^{32} \cdot 11^4 \equiv 3 \cdot 9 \cdot 3 \pmod{13}.$$

$$3 \cdot 9 = 27 \equiv 1 \pmod{13}, \quad 1 \cdot 3 = 3.$$

Hence, $50^{100} \equiv 3 \pmod{13}$.

□