King Saud University, College of Science
Department of Mathematics
Second Semester of 1447 H - (2025 – 2026 G)

First Midterm Exam of Math 431.

Let $G = (V, E)$ be a simple finite graph of order $n \geq 2$, and fix a vertex $x \in V$. Denote by $\overline{G} = (V, \overline{E})$ the complement of G, and by $d_G(x)$ the degree of x in G.

Question 1

1. How many edges does a graph have if the sum of the vertex degrees is 48?
2. How many edges does a 2-regular graph on 14 vertices have?
3. A graph has 47 edges. What is the minimum possible number of vertices?
4. How many edges does the complement of $K_{2,3}$ have?
5. Let n and m be positive integers. How many edges does the complement of $K_{n,m}$ have?

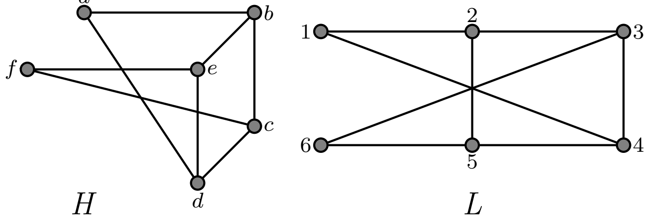
Question 2

1. Give an example of a self-complementary graph of order 4, and another of order 5.
2. (a) Prove that if there exists $x \in V$ such that $d_G(x) = 0$, then G is not a self-complementary graph.
   (b) Prove that if G is a self-complementary graph, then for every $x \in V$ we have $1 \leq d_G(x) \leq n - 2$.
   (c) Prove that if there exists $x \in V$ such that $d_G(x) = p$ with $1 \leq p \leq n - 2$, then there exists $y \in V$ such that $d_G(y) = n - 1 - p$.
3. Determine all values of n for which the path P_n (respectively, the cycle C_n) is self-complementary.

Question 3

Let H and L be two simple graphs (given in the figure).

1. Give the adjacency matrix of L, determine the degree of each vertex of L and determine the number of edges of L.
2. Determine whether L is a bipartite graph, and justify your answer.
3. Determine whether K and L are isomorphic, and justify your answer.



Question 4

Consider the graph $G = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}\})$

1. Find all walks from v_1 to v_4 of length 4.
2. Find all paths from v_1 to v_4 of length 4.

Solutions

Question 1

1. By the Handshaking Lemma, $\sum \deg(v) = 2|E|$.

   Thus, $2|E| = 48 \Rightarrow |E| = 24$.

2. In a 2-regular graph, each vertex has degree 2. Sum of degrees: $14 \times 2 = 28 = 2|E|$.

   Hence, $|E| = 14$.

3. The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$. We need $\frac{n(n-1)}{2} \geq 47$.
   First method: Testing: $n = 9 \Rightarrow 36$ edges (not enough)
   $n = 10 \Rightarrow 45$ edges (not enough)
   $n = 11 \Rightarrow 55$ edges (enough).

   Minimum number of vertices: $\boxed{n = 11}$.

   Second method: The inequality
   $$\frac{n(n-1)}{2} \geq 47$$
   is equivalent to
   $$n^2 - n - 94 \geq 0.$$

   Solving the quadratic inequality
   $$n^2 - n - 94 \geq 0,$$
   we compute the discriminant:
   $$\Delta = (-1)^2 + 4 \cdot 94 = 377$$
   Since n must be a positive integer. Hence,
   $$n \geq \frac{1 + \sqrt{377}}{2}.$$
   Therefore, the smallest possible value is $\boxed{n = 11}$.
   We can obtain such a graph by starting from the complete graph K_{11} and deleting $55 - 47 = 8$ edges. Hence,
   $$G = K_{11} \setminus \{e_1, \ldots, e_8\}.$$

4. The graph $K_{2,3}$ has $2 \times 3 = 6$ edges. Total vertices: 5.

   A complete graph on 5 vertices has $\binom{5}{2} = 10$ edges.

   Complement edges: $10 - 6 = 4$.

5. Total vertices in the complete bipartite graph $K_{n,m}$: $n + m$.
   Edges in the complete graph K_{n+m}: $\binom{n+m}{2}$.
   Edges in $K_{n,m}$: nm.
   Thus the complement of $K_{n,m}$ has the number of edges
   $\binom{n+m}{2} - nm = \frac{(n+m)(n+m-1)}{2} - nm = \frac{n(n-1)}{2} + \frac{m(m-1)}{2}$.

Question 2

1. **Examples.**

   A self-complementary graph of order 4 is the path P_4. Indeed, $\overline{P_4} \cong P_4$.

   A self-complementary graph of order 5 is the cycle C_5. Indeed, $\overline{C_5} \cong C_5$.

2. (a) Suppose there exists $x \in V$ such that $d_G(x) = 0$. Then x is adjacent to every other vertex in $\overline{G}$, so $d_{\overline{G}}(x) = n - 1$.
   If G were self-complementary, then $G \cong \overline{G}$ and the degree sequences would coincide. But G has a vertex of degree 0 while $\overline{G}$ has a vertex of degree $n-1$, which is impossible for isomorphic graphs. Hence G is not self-complementary.

   (b) If G is self-complementary, then $G \cong \overline{G}$. Thus no vertex can have degree 0 or $n-1$ by part (a). Therefore for every $x \in V$,
   $$1 \leq d_G(x) \leq n - 2.$$

   (c) Let G be self-complementary and suppose there exists $x \in V$ such that $d_G(x) = p$. Since $G \cong \overline{G}$, there exists a vertex $y \in V$ whose degree in G equals the degree of x in $\overline{G}$. But
   $$d_{\overline{G}}(x) = n - 1 - d_G(x) = n - 1 - p.$$
   Hence there exists $y \in V$ such that
   $$d_G(y) = n - 1 - p.$$

3. **Order of a self-complementary graph.**

   If G is self-complementary of order n, $n \in \mathbb{N}$, then
   $$|E(G)| = |\overline{E(G)}|.$$
   But
   $$|E(G)| + |\overline{E(G)}| = \binom{n}{2}.$$
   Hence
   $$2|E(G)| = \binom{n}{2} = \frac{n(n-1)}{2}, \quad \text{so} \quad |E(G)| = \frac{n(n-1)}{4}.$$
   Therefore $n(n-1)$ must be divisible by 4. This happens if and only if
   $$n \equiv 0 \pmod 4 \quad \text{or} \quad n \equiv 1 \pmod 4.$$
   Thus
   $$n = 4p \quad \text{or} \quad n = 4p + 1$$
   for some $p \in \mathbb{N}$.

   Paths and cycles.

   For paths:
   $$|E(P_n)| = n - 1.$$
   If P_n is self-complementary, then
   $$n - 1 = \frac{n(n-1)}{4}.$$
   Hence $n = 1$ or $n = 4$, since P_1 and P_4 are self complementary.
   Therefor P_n is self-complementary if and only if $n = 1$, $n = 4$.

   For cycles:
   $$|E(C_n)| = n.$$
   If C_n is self-complementary, then
   $$n = \frac{n(n-1)}{4}.$$
   Hence $n = 5$. Since, C_5 is the self-complementary graph.

   Therefore, C_n is self-complementary if and only if $n = 5$.

Question 3

Solution

Let L be the graph with vertex set

$$V(L) = \{1, 2, 3, 4, 5, 6\}.$$

From the figure we read the edges:

$$E(L) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{2, 5\}, \{3, 6\}, \{1, 4\}\}.$$

1. Adjacency matrix, degrees, and number of edges

We order the vertices as $(1, 2, 3, 4, 5, 6)$.

The adjacency matrix $A(L) = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E(L), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$A(L) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Degrees of vertices.

$$\deg(1) = 2, \quad \deg(2) = 3, \quad \deg(3) = 3, \quad \deg(4) = 3, \quad \deg(5) = 3, \quad \deg(6) = 2.$$

Hence the degree sequence is

$$(3, 3, 3, 3, 2, 2).$$

Number of edges.
Using the Handshaking Lemma,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Thus

$$2 + 3 + 3 + 3 + 3 + 2 = 16 = 2|E|,$$

so

$$|E(L)| = 8.$$

2. Is L bipartite?

Consider the subsets of $V(L)$ defined by

$$X = \{1, 3, 5\} \qquad \text{and} \qquad Y = \{2, 4, 6\}.$$

We claim that (X, Y) is a bipartition of L. Indeed, by inspection of the edge set of L, every edge has one endpoint in X and the other in Y. In particular, there are no edges with both endpoints in X, and no edges with both endpoints in Y.

Therefore by using the definition of bipartite graph, L is bipartite, with bipartition

$$V(L) = X \cup Y.$$

3. Are H and L isomorphic?

First compare invariants.

Both graphs have:

$$|V| = 6, \qquad |E| = 8.$$

The degree sequence of L is

$$(3, 3, 3, 3, 2, 2).$$

From the figure of H, one verifies that H has the same degree sequence.

Moreover, both graphs are bipartite and admit the same structural pattern: four vertices of degree 3 forming the "core" and two vertices of degree 2 attached symmetrically.

We can explicitly define an isomorphism by matching:

$$a \leftrightarrow 1, \quad b \leftrightarrow 2, \quad c \leftrightarrow 3, \quad d \leftrightarrow 4, \quad e \leftrightarrow 5, \quad f \leftrightarrow 6.$$

By the adjacency matrix $A(H)$ of the graph H, we observe that $A(L) = A(H)$. One checks that adjacency is preserved.

Conclusion.
The graphs H and L are isomorphic.

Question 4

Let $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5\}$.

First method Alternative solution using the adjacency matrix:

Recall that the number of walks of length k from v_i to v_j is equal to the (i, j)-entry of A^k, where A is the adjacency matrix of the graph.

Ordering the vertices as $(v_1, v_2, v_3, v_4, v_5)$, the adjacency matrix of G is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We compute A^2 first. Since

$$A^4 = A^2A^2,$$

the entry $(A^4)_{1,4}$ is obtained by multiplying the first row of A^2 by the fourth column of A^2.

A direct computation gives

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 4 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

Thus,

$$(A^4)_{1,4} = (\text{row } 1 \text{ of } A^2) \cdot (\text{column } 4 \text{ of } A^2).$$

That is,

$$(1, 0, 1, 1, 1) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = 1 + 0 + 1 + 2 + 2 = 6.$$

Therefore,

$$(A^4)_{1,4} = 6,$$

which is the number of walks of length 4 from v_1 to v_4.

Conclusion.

There are exactly 6 walks of length 4 from v_1 to v_4.

Second method: Observe that v_1 is adjacent only to v_2.

1. Walks from v_1 to v_4 of length 4

A walk of length 4 has the form

$$v_1, x_1, x_2, x_3, v_4,$$

where consecutive vertices are adjacent.
Since v_1 is adjacent only to v_2, every such walk must begin

$$v_1 \to v_2.$$

We now enumerate all possible continuations of length 3 from v_2 to v_4.

Step 1: Possible first moves from v_2.
The neighbours of v_2 are

$$v_1, v_3, v_4, v_5.$$

We consider each possibility.

Case 1: $v_1 \to v_2 \to v_1$.
From v_1 we must return to v_2, and then go to v_4:

$$v_1, v_2, v_1, v_2, v_4.$$

Case 2: $v_1 \to v_2 \to v_3$.
From v_3 we may go to v_2, v_4, or v_5.

$$\begin{aligned} &v_1, v_2, v_3, v_2, v_4, \\ &v_1, v_2, v_3, v_4, v_3, \quad \text{(invalid, does not end at } v_4\text{)}, \\ &v_1, v_2, v_3, v_5, v_3, \quad \text{(invalid)}. \end{aligned}$$

The only valid one ending at v_4 is

$$v_1, v_2, v_3, v_2, v_4.$$

Case 3: $v_1 \to v_2 \to v_4$.
From v_4 we may go to v_2 or v_3.

$$\begin{aligned} &v_1, v_2, v_4, v_2, v_4, \\ &v_1, v_2, v_4, v_3, v_4. \end{aligned}$$

Both end at v_4 and are valid walks.
Case 4: $v_1 \to v_2 \to v_5$.
From v_5 we may go to v_2 or v_3.

$$\begin{aligned} &v_1, v_2, v_5, v_2, v_4, \\ &v_1, v_2, v_5, v_3, v_4. \end{aligned}$$

Both are valid.
After calculating the number of walks using two methods, we provide this list of walks.
All walks of length 4 from v_1 to v_4:

1. v_1, v_2, v_1, v_2, v_4
2. v_1, v_2, v_3, v_2, v_4
3. v_1, v_2, v_4, v_2, v_4
4. v_1, v_2, v_4, v_3, v_4
5. v_1, v_2, v_5, v_2, v_4
6. v_1, v_2, v_5, v_3, v_4

There are therefore **6 walks**.

2. Paths from v_1 to v_4 of length 4

A path cannot repeat vertices.

We inspect the six walks above:

1. v_1, v_2, v_1, v_2, v_4 (repeats)
2. v_1, v_2, v_3, v_2, v_4 (repeats v_2)
3. v_1, v_2, v_4, v_2, v_4 (repeats)
4. v_1, v_2, v_4, v_3, v_4 (repeats v_4)
5. v_1, v_2, v_5, v_2, v_4 (repeats v_2)
6. v_1, v_2, v_5, v_3, v_4 (all vertices distinct).

Thus the only path of length 4 from v_1 to v_4 is

$$v_1, v_2, v_5, v_3, v_4.$$

Conclusion.

$$\text{Number of walks} = 6, \qquad \text{Number of paths} = 1.$$