Midterm Exam 1 Solution

Department of Mathematics, College of Science

Course: Math 209

Question 1 [6 points]

Determine whether the sequence converges or diverges.

1. $a_n = (-1)^n$

   This sequence alternates between 1 and -1, so it does not approach a single number.
   Answer: Diverges.

2. $a_n = \dfrac{(-1)^{n^2}}{n}$

   Since $(-1)^{n^2} = \pm 1$, we have

   $$\left|\frac{(-1)^{n^2}}{n}\right| \le \frac{1}{n} \to 0.$$

   By squeeze theorem

   $$a_n \to 0.$$

   Answer: Converges to 0.

3. $a_n = \sqrt{n^2+2} - \sqrt{n^2+1}$

   Rationalize:

   $$a_n = \frac{(n^2+2)-(n^2+1)}{\sqrt{n^2+2}+\sqrt{n^2+1}} = \frac{1}{\sqrt{n^2+2}+\sqrt{n^2+1}}.$$

   As $n \to \infty$, the denominator $\to \infty$, so

   $$a_n \to 0.$$

   Answer: Converges to 0.

Question 2 [3 points]

For each statement, write **True or False**.

1. If $\lim a_n = 0$, then $\sum a_n$ converges.

   This is not always true. Example: $\sum \frac{1}{n}$ diverges although $\frac{1}{n} \to 0$.
   Answer: False.

2. If $\sum a_n$ converges absolutely, then it converges.

   Absolute convergence implies convergence.
   Answer: True.

3. If $0 \leq a_n \leq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

   This is not true. Example: $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $0 \leq a_n \leq b_n$, $\sum b_n$ diverges, but $\sum a_n$ converges.
   Answer: False.

Question 3 [9 points]

For each series, choose an appropriate test and justify briefly.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   Use the Integral Test. Since
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx$$
   diverges, the series also diverges.
   Answer: Diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

   Compare with $\frac{1}{n}$:
$$\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n + \sqrt{n}} = 1.$$
   Since $\sum \frac{1}{n}$ diverges, by Limit Comparison Test,
$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$
   diverges.
   Answer: Diverges.

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

   For an alternating series, we need $\frac{n}{n+1} \to 0$. But
$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$
   So the series fails the nth-term test.
   Answer: Diverges.

Question 4 [7 points]

Let
$$f(x) = \sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{n+1}.$$

1. Find the radius of convergence.

   Write the series as

$$\sum_{n=0}^{\infty} \frac{(3(x+1))^n}{n+1}.$$

   Using the Root Test or Ratio Test, convergence requires

$$|3(x+1)| < 1.$$

   Thus

$$|x+1| < \frac{1}{3}.$$

   So the radius of convergence is

$$\boxed{R = \frac{1}{3}.}$$

2. Find the interval of convergence.

   From

$$|x+1| < \frac{1}{3},$$

   we get

$$-\frac{1}{3} < x+1 < \frac{1}{3}.$$

   Subtracting 1:

$$-\frac{4}{3} < x < -\frac{2}{3}.$$

   So the possible interval is

$$\left[-\frac{4}{3}, -\frac{2}{3}\right]$$

   with endpoints to be checked.

3. Determine whether the series converges at the endpoints.

   At $x = -\frac{4}{3}$:

$$x+1 = -\frac{1}{3} \quad \Rightarrow \quad 3(x+1) = -1.$$

   So the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

   This is the alternating harmonic series, which converges.

   At $x = -\frac{2}{3}$:

$$x+1 = \frac{1}{3} \quad \Rightarrow \quad 3(x+1) = 1.$$

   So the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n+1},$$

   which is the harmonic series, and it diverges.

   Therefore the interval of convergence is

$$\boxed{\left[-\frac{4}{3}, -\frac{2}{3}\right)}.$$

Final Answers Summary

- Sequences:
  - $(-1)^n$: diverges
  - $\dfrac{(-1)^{n^2}}{n}$: converges to 0
  - $\sqrt{n^2+2} - \sqrt{n^2+1}$: converges to 0
- True/False:
  - False
  - True
  - False
- Series:
  - $\sum \dfrac{1}{n \ln n}$: diverges
  - $\sum \dfrac{1}{n+\sqrt{n}}$: diverges
  - $\sum (-1)^n \dfrac{n}{n+1}$: diverges
- Power series:
  - Radius: $\dfrac{1}{3}$
  - Interval: $\left[-\dfrac{4}{3}, -\dfrac{2}{3}\right)$