MIDTERM 2 REVIEW

Previous Midterm 2 Exams

Semesters 471 and 472 · Sections 4.1 – 4.8 and 6.1 – 6.3

MATH 244 — Linear Algebra
Midterm 2 · Sunday 15 November 2026
King Saud University · Department of Mathematics

SEMESTER 471 · MIDTERM 2

Midterm Exam 2, Semester 471

25 marks · 1½ hours · four questions. Work each question on paper first, then go through its parts one slide at a time.

1. Question 1 — three multiple-choice items and two examples (3 + 2 marks)
2. Question 2 — bases for $\mathrm{row}(A)$ and $\mathrm{col}(A)$; relations among the columns (4 + 3 marks)
3. Question 3 — polynomials in P_3; a null space spanned by part of a basis (3 + 4 marks)
4. Question 4 — a transition matrix; the angle between two matrices (3 + 3 marks)

Select the correct choice; give examples

I. Select the correct choice: (3 marks)

(i) If $W = \{(1,1), (1,2)\}$, then W is:

a) a vector space **b)** a subspace of $\mathbb{R}^2$ **c)** a basis of $\mathbb{R}^2$ **d)** linearly dependent

(ii) If $B = \{u, v, w\}$ is an orthogonal set of non-zero vectors in an inner product space E and $\alpha u + \beta v + \gamma w = 0$ for some scalars α, β and γ, then:

a) $\alpha = \beta = \gamma = 0$ **b)** $u = v = w$ **c)** $\alpha = \gamma,\ v = 0,\ u = w$ **d)** B is linearly dependent

(iii) If A is an invertible matrix of order 3, then $\mathrm{rank}(A)$ is equal to:

a) 0 **b)** 1 **c)** 2 **d)** 3

II. Give an example for each of the following. You don't have to prove your answers. (2 marks)

(i) A linearly dependent subset of $\mathbb{R}^3$ consisting of three vectors in which every two vectors are linearly independent.

(ii) A matrix having rank 3 and nullity 4.

A basis is a set of vectors, not a space

(i) If $W = \{(1,1), (1,2)\}$, then W is:

a) a vector space **b)** a subspace of $\mathbb{R}^2$ **c)** a basis of $\mathbb{R}^2$ **d)** linearly dependent

SOLUTION **ANSWER (c)**

Neither vector is a multiple of the other, so W is linearly independent. The determinant of the matrix with these vectors as columns confirms it:

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

Two linearly independent vectors in $\mathbb{R}^2$, a space of dimension 2, form a basis of $\mathbb{R}^2$.

(a) and (b) fail: W holds only two vectors. It does not contain $(0,0)$, and it is not closed under addition: $(1,1) + (1,2) = (2,3) \notin W$. The space is $\text{span}(W) = \mathbb{R}^2$; the set W is a basis of it.

Orthogonal nonzero vectors are independent

(ii) If $B = \{u, v, w\}$ is an orthogonal set of non-zero vectors in an inner product space E and $\alpha u + \beta v + \gamma w = 0$ for some scalars α, β and γ, then:

a) $\alpha = \beta = \gamma = 0$ **b)** $u = v = w$ **c)** $\alpha = \gamma,\ v = 0,\ u = w$ **d)** B is linearly dependent

SOLUTION

ANSWER (a)

Take the inner product of both sides with u. Orthogonality gives $\langle v, u\rangle = \langle w, u\rangle = 0$, so

$$0 = \langle 0, u\rangle = \langle \alpha u + \beta v + \gamma w,\ u\rangle = \alpha\langle u, u\rangle + \beta\langle v, u\rangle + \gamma\langle w, u\rangle = \alpha \parallel u \parallel^2$$

Since $u \neq 0$, $\parallel u \parallel^2 > 0$, hence $\alpha = 0$. The inner product with v, then with w, gives $\beta = 0$ and $\gamma = 0$ in the same way.

An orthogonal set of nonzero vectors is always linearly independent, so (d) is false. (b) and (c) are impossible: the vectors are nonzero, and a nonzero vector is not orthogonal to itself.

Invertible means full rank

(iii) If A is an invertible matrix of order 3, then $\text{rank}(A)$ is equal to:

a) 0 **b)** 1 **c)** 2 **d)** 3

SOLUTION

ANSWER (d)

A is invertible, so $Ax = 0$ has only the trivial solution: $\text{nullity}(A) = 0$. By the rank–nullity theorem, with 3 columns,

$$\text{rank}(A) = 3 - \text{nullity}(A) = 3 - 0 = 3$$

Equivalently, $\text{RREF}(A) = I_3$, which has a pivot in each of its 3 columns.

(a) Rank 0 belongs to the zero matrix only. (b), (c) A rank below 3 leaves a free variable, so $Ax = 0$ would have nontrivial solutions and A could not be invertible.

Dependent, yet every two are independent

> **II.** Give an example for each of the following. You don't have to prove your answers.
>
> **(i)** A linearly dependent subset of $\mathbb{R}^3$ consisting of three vectors in which every two vectors are linearly independent.

SOLUTION

Three vectors in one plane through the origin, no two of them on the same line. The key's example:

$$\{(1,0,0),\ (0,1,0),\ (1,1,0)\}$$

- **Dependent:** $(1,1,0) = (1,0,0) + (0,1,0)$, that is, $1 \cdot (1,0,0) + 1 \cdot (0,1,0) - 1 \cdot (1,1,0) = 0$.
- **Every two independent:** two vectors are dependent only when one is a multiple of the other, and no vector here is a multiple of another.

> Recipe: take two vectors u, v that are not multiples of each other, and add $u + v$ as the third vector.

Rank 3 and nullity 4: count the columns

> **II.** Give an example for each of the following. You don't have to prove your answers.
>
> **(ii)** A matrix having rank 3 and nullity 4.

SOLUTION

Rank + nullity = number of columns, so the matrix has $3 + 4 = 7$ columns, and it needs at least 3 rows to have rank 3. The key's example is 3×7:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is already in reduced row echelon form: 3 pivots give rank 3, and the 4 columns without a pivot give 4 free variables, so nullity 4.

> Rank–nullity: $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$, the number of columns. Any $m\times7$ matrix with 3 pivots works; no matrix with 4, 5 or 6 columns can.

SEMESTER 471 · QUESTION 2 · 4 + 3 MARKS

Row space, column space and the columns of A

Let $A = \begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & 3 & 0 & -1 & -6 \\ 3 & 4 & 2 & -4 & -1 \end{bmatrix}$. Then:

(i) Find bases for $\mathrm{row}(A)$ and $\mathrm{col}(A)$. (4 marks)

(ii) Express the last three columns of the matrix A as linear combinations of the first two. (3 marks)

Bases for $\mathrm{row}(A)$ and $\mathrm{col}(A)$

(i) Find bases for $\mathrm{row}(A)$ and $\mathrm{col}(A)$.

$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & 3 & 0 & -1 & -6 \\ 3 & 4 & 2 & -4 & -1 \end{bmatrix}$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & 3 & 0 & -1 & -6 \\ 3 & 4 & 2 & -4 & -1 \end{bmatrix} \overset{A_{1,2}^{(-2)},\, A_{1,3}^{(-3)}}{\rightarrow} \begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 0 & 1 & -4 & 5 & -16 \\ 0 & 1 & -4 & 5 & -16 \end{bmatrix} \overset{A_{2,1}^{(-1)},\, A_{2,3}^{(-1)}}{\rightarrow} \begin{bmatrix} 1 & 0 & 6 & -8 & 21 \\ 0 & 1 & -4 & 5 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{RREF}(A)$$

Basis for $\mathrm{row}(A)$: nonzero rows of the RREF

$$\{(1,0,6,-8,21),\ (0,1,-4,5,-16)\}$$

Basis for $\mathrm{col}(A)$: pivot columns of A itself

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}\right\}$$

Row operations keep the row space but change the column space: $(1,0,0)^T$ is a pivot column of the RREF, yet it is not in $\mathrm{col}(A)$, where $x_3 = x_1 + x_2$.

Reading column relations off the RREF

(ii) Express the last three columns of the matrix A as linear combinations of the first two.

$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & 3 & 0 & -1 & -6 \\ 3 & 4 & 2 & -4 & -1 \end{bmatrix}$$

SOLUTION

Row operations keep every linear relation among the columns, so read the relations off $\mathrm{RREF}(A)$:

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 6 & -8 & 21 \\ 0 & 1 & -4 & 5 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ -1 \\ -4 \end{bmatrix} = -8\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix} = 21\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 16\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Check the first one: $6(1,2,3) - 4(1,3,4) = (6-4,\ 12-12,\ 18-16) = (2,0,2)$, the third column of A.

Polynomials, and a null space from a basis

(i) Let $F = \{p = 2 - x,\ q = 5 + 3x - x^2 + 2x^3,\ r = 1 + 7x + x^2 + 4x^3\}$. Find the real numbers α, β, γ so that $-3 + 5x^2 + 2x^3 = \alpha p + \beta q + \gamma r$. Also, show that the set F is linearly independent in the vector space P_3. (3 marks)

(ii) Let $G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be a basis of the vector space $\mathbb{R}^6$. Let A be a 4×6 matrix such that $A(v_1)$ and $A(v_2)$ are linearly independent vectors in $\mathbb{R}^4$ and $A(v_i) = 0$, for all $i = 3,4,5,6$. Show that the null space $N(A) = \text{span}(\{v_3, v_4, v_5, v_6\})$. Also, find $\text{nullity}(A^T)$. (4 marks)

A combination in P_3, and independence of F

(i) Let $F = \{p = 2 - x,\ q = 5 + 3x - x^2 + 2x^3,\ r = 1 + 7x + x^2 + 4x^3\}$. Find the real numbers α, β, γ so that $-3 + 5x^2 + 2x^3 = \alpha p + \beta q + \gamma r$. Also, show that the set F is linearly independent in the vector space P_3.

SOLUTION

Expand $\alpha p + \beta q + \gamma r$ and match the coefficients of 1, x, x^2, x^3 with those of $-3 + 5x^2 + 2x^3$:

$$\begin{array}{rcrcrcr} 2\alpha & + & 5\beta & + & \gamma & = & -3 \\ -\alpha & + & 3\beta & + & 7\gamma & = & 0 \\ & & -\beta & + & \gamma & = & 5 \\ & & 2\beta & + & 4\gamma & = & 2 \end{array}$$

The last two equations have no α: $\gamma = \beta + 5$, so $2\beta + 4(\beta + 5) = 2$, giving $\beta = -3$ and $\gamma = 2$.

The first equation: $2\alpha - 15 + 2 = -3$, so $\alpha = 5$. The second checks: $-5 - 9 + 14 = 0$.

Independence. Replace the right-hand sides by 0. The same steps give $\gamma = \beta$, then $6\beta = 0$, so $\beta = \gamma = 0$ and $2\alpha = 0$: only the trivial solution.

$-3 + 5x^2 + 2x^3 = 5p - 3q + 2r$, and F is linearly independent in P_3.

Which vectors does A send to zero?

(ii) Let $G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be a basis of the vector space $\mathbb{R}^6$. Let A be a 4×6 matrix such that $A(v_1)$ and $A(v_2)$ are linearly independent vectors in $\mathbb{R}^4$ and $A(v_i) = 0$, for all $i = 3,4,5,6$. Show that the null space $N(A) = \text{span}(\{v_3, v_4, v_5, v_6\})$. Also, find $\text{nullity}(A^T)$.

SOLUTION

First $\text{span}\{v_3, \dots, v_6\} \subseteq N(A)$: each of $v_3, \dots, v_6$ lies in $N(A)$, and the subspace $N(A)$ contains their span.

Then $N(A) \subseteq \text{span}\{v_3, \dots, v_6\}$: let $u \in N(A)$. As G is a basis, $u = \alpha_1 v_1 + \cdots + \alpha_6 v_6$, and $Av_i = 0$ for $i \geq 3$, so

$$0 = Au = \alpha_1 Av_1 + \alpha_2 Av_2 + \alpha_3 Av_3 + \cdots + \alpha_6 Av_6 = \alpha_1 Av_1 + \alpha_2 Av_2$$

Av_1, Av_2 are independent, so $\alpha_1 = \alpha_2 = 0$ and $u = \alpha_3 v_3 + \cdots + \alpha_6 v_6 \in \text{span}\{v_3, \dots, v_6\}$.

Nullity of A^T. $\{v_3, \dots, v_6\}$ is part of a basis, hence independent: a basis of $N(A)$, and $\text{nullity}(A) = 4$. A^T is 6×4:

$$\text{nullity}(A^T) = 4 - \text{rank}(A^T) = 4 - \text{rank}(A) = 4 - (6 - 4) = 2$$

A transition matrix; the angle between two matrices

(i) Consider a vector space E of dimension 3. Let $B = \{u_1, u_2, u_3\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for E such that the transition matrix ${}_C P_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ from B to C. Compute the coordinate vector $[v_1 - 2v_2 + 3v_3]_B$. (3 marks)

(ii) Consider the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in the inner product space $M_2(\mathbb{R})$, consisting of all real matrices of size 2, with inner product $\langle A, B \rangle = \text{trace}\left(AB^T\right)$. Find the angle between the matrices A and B. Also, verify the Pythagorean theorem for the same matrices. (3 marks)

From ${}_CP_B$ to ${}_BP_C$

(i) Consider a vector space E of dimension 3. Let $B = \{u_1, u_2, u_3\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for E such that the transition matrix ${}_CP_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ from B to C. Compute the coordinate vector $[v_1 - 2v_2 + 3v_3]_B$.

SOLUTION

We need the reverse matrix ${}_BP_C = ({}_CP_B)^{-1}$. Reduce $[\,{}_CP_B \mid I\,]$ to $[\,I \mid {}_BP_C\,]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-1)},\, A_{1,3}^{(-1)},\, I_{2,3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{M_2^{(-1)},\, M_3^{(-1)},\, A_{2,1}^{(-1)},\, A_{3,1}^{(-1)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

The vector is written in terms of C, so $[v_1 - 2v_2 + 3v_3]_C = (1, -2, 3)^T$ and

$$[v_1 - 2v_2 + 3v_3]_B = {}_BP_C\,[v_1 - 2v_2 + 3v_3]_C = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Check: ${}_CP_B\,(0, -2, 3)^T = (1, -2, 3)^T$, so $v_1 - 2v_2 + 3v_3 = -2u_2 + 3u_3$.

The angle between two matrices

(ii) Consider the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in the inner product space $M_2(\mathbb{R})$, consisting of all real matrices of size 2, with inner product $\langle A, B \rangle = \text{trace}\left(AB^T\right)$. Find the angle between the matrices A and B. Also, verify the Pythagorean theorem for the same matrices.

SOLUTION

$$\langle A, B \rangle = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^T\right) = \text{trace}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = 2 - 2 = 0$$

So $\cos\theta = \langle A, B \rangle / (\| A \| \| B \|) = 0$ and $\theta = \pi/2$: A and B are orthogonal in $M_2(\mathbb{R})$.

Pythagoras. With $A + B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $\| X \|^2 = \langle X, X \rangle = \text{trace}\left(XX^T\right)$:

$$\| A \|^2 = \text{trace}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = 5, \quad \| B \|^2 = \text{trace}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = 5, \quad \| A + B \|^2 = \text{trace}\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = 10$$

$\| A + B \|^2 = 10 = 5 + 5 = \| A \|^2 + \| B \|^2$, as the Pythagorean theorem says for orthogonal A and B.

SEMESTER 472 · MIDTERM 2

Midterm Exam 2, Semester 472

25 marks · 1½ hours · calculators not allowed. Work each question on paper first, then go through its parts one slide at a time.

1. Question 1 — ten multiple-choice items (10 × 1 mark)
2. Question 2 — a dependent spanning set; testing, reducing and extending bases (2 + 2 + 3 marks)
3. Question 3 — coordinates and a transition matrix; rank and nullity ((2 + 3) + 3 marks)

Which of the given choices are correct?

(i) If $B = \{u, v, w\}$ is a basis of $\mathbb{R}^3$ and $ru + sv + tw = 0$ for some scalars r, s, t, then:

a) $r = u,\ s = v,\ t = w$ **b)** $r = s = t = 0$ **c)** $u = v = w$ **d)** B is linearly dependent

(ii) If a subset S of the vector space $\mathbb{R}^3$ spans $\mathbb{R}^3$ what must be true?

a) All vectors in S are non-zero.

b) S contains exactly three vectors.

c) S is linearly independent.

d) S contains at least three linearly independent vectors.

(iii) If A is a matrix of size 5×7 then its:

a) rank is 5 **b)** rank is 7 **c)** rows are linearly independent **d)** columns are linearly dependent

Which of the given choices are correct?

(iv) If $S = \{u, v, w\}$ is an ordered basis of a vector space V, then the coordinate vector $[u]_S$ with respect to the ordered basis S is:

a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(v) If $S = \{u, v, w, x\}$ is a subset of the vector space $\mathbb{R}^3$, then S must be:

a) a basis of $\mathbb{R}^3$ **b)** a subspace of $\mathbb{R}^3$ **c)** linearly dependent **d)** linearly independent

(vi) Which of the following sets is a subspace of the vector space $\mathbb{R}^3$?

a) $\{(x, y, z) : x > 0\}$

b) $\{(x, y, z) : x + y + z = 1\}$

c) $\{(x, y, z) : x + y + z = 0\}$

d) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

Which of the given choices are correct?

(vii) If a subset $S \neq \{0\}$ of a vector space V is linearly dependent, then:

a) The set S must span the vector space V.

b) No vector in S can be written as a linear combination of other vectors in S.

c) At least one vector in S is a linear combination of other vectors in S.

d) One vector in S must be 0.

(viii) Any transition matrix P must satisfy that:

a) P is diagonal **b)** $\det(P) = 0$ **c)** P is invertible **d)** P is symmetric

(ix) If A is a 3×6 matrix with linearly independent rows, then:

a) $\text{rank}(A) = 6$ **b)** $\text{rank}(A) = 3$ **c)** $\text{rank}(A) = 4$ **d)** $\text{rank}(A) = 5$

(x) If a matrix A is of size 4×6 such that $\dim\big(\text{col}(A)\big) = 3$ then $\dim\big(N(A)\big)$ is equal to:

a) ≤ 2 **b)** 1 **c)** 3 **d)** 0

A basis allows only the trivial combination

(i) If $B = \{u, v, w\}$ is a basis of $\mathbb{R}^3$ and $ru + sv + tw = 0$ for some scalars r, s, t, then:

a) $r = u,\ s = v,\ t = w$ **b)** $r = s = t = 0$ **c)** $u = v = w$ **d)** B is linearly dependent

SOLUTION **ANSWER (b)**

A basis is linearly independent. By the definition of independence, the only combination of u, v, w equal to 0 is the one with every scalar zero:

$$ru + sv + tw = 0 \quad \Rightarrow \quad r = s = t = 0$$

(a) equates scalars with vectors, which has no meaning. (c) The three vectors of a basis are distinct. (d) contradicts the independence of a basis.

SEMESTER 472 · QUESTION 1 (ii) · 1 MARK

What a spanning set must contain

(ii) If a subset S of the vector space $\mathbb{R}^3$ spans $\mathbb{R}^3$ what must be true?

a) All vectors in S are non-zero.

b) S contains exactly three vectors.

c) S is linearly independent.

d) S contains at least three linearly independent vectors.

SOLUTION

ANSWER (d)

Every spanning set of a vector space contains a basis of that space, and every basis of $\mathbb{R}^3$ has exactly 3 vectors. So S contains 3 linearly independent vectors.

(a) fails: $\{(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}$ spans $\mathbb{R}^3$ and contains the zero vector.

(b), (c) fail: $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ spans $\mathbb{R}^3$, has four vectors and is dependent.

In $\mathbb{R}^n$: a spanning set has at least n vectors, an independent set has at most n, and a basis has exactly n.

Seven columns in $\mathbb{R}^5$

(iii) If A is a matrix of size 5×7 then its:

a) rank is 5 **b)** rank is 7 **c)** rows are linearly independent **d)** columns are linearly dependent

SOLUTION **ANSWER (d)**

The 7 columns of A are vectors in $\mathbb{R}^5$, and more than 5 vectors in a space of dimension 5 are linearly dependent. In terms of rank:

$$\operatorname{rank}(A) \le \min(5,7) = 5, \qquad \operatorname{nullity}(A) = 7 - \operatorname{rank}(A) \ge 2$$

So $Ax = 0$ has nontrivial solutions, and each one is a nontrivial relation among the columns.

(a) and (c) can fail: the 5×7 zero matrix has rank 0 and dependent rows. (b) is impossible: the rank is at most 5.

The coordinates of a basis vector

(iv) If $S = \{u, v, w\}$ is an ordered basis of a vector space V, then the coordinate vector $[u]_S$ with respect to the ordered basis S is:

a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

SOLUTION **ANSWER (c)**

$$u = 1 \cdot u + 0 \cdot v + 0 \cdot w \quad \Rightarrow \quad [u]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Order matters: u is the first vector of the ordered basis S, so its 1 sits in the first entry. Coordinates relative to a basis are unique, so this is the only answer.

(a) is $[w]_S$, the third basis vector. (b) is the coordinate vector of 0, and (d) is $[u + v + w]_S$.

SEMESTER 472 · QUESTION 1 (v) · 1 MARK

Four vectors in $\mathbb{R}^3$

(v) If $S = \{u, v, w, x\}$ is a subset of the vector space $\mathbb{R}^3$, then S must be:

a) a basis of $\mathbb{R}^3$ **b)** a subspace of $\mathbb{R}^3$ **c)** linearly dependent **d)** linearly independent

SOLUTION **ANSWER (c)**

$\dim\mathbb{R}^3 = 3$, and any set of more than 3 vectors in $\mathbb{R}^3$ is linearly dependent. S has four vectors, so it is dependent.

Directly: $c_1u + c_2v + c_3w + c_4x = 0$ is a homogeneous system of 3 equations in 4 unknowns. It has a free variable, hence a nontrivial solution.

(a) A basis of $\mathbb{R}^3$ has exactly 3 vectors. (b) A set of four vectors is not closed under scalar multiplication, so it is not a subspace. (d) is the opposite of the truth.

Which set is a subspace?

(vi) Which of the following sets is a subspace of the vector space $\mathbb{R}^3$?

a) $\{(x,y,z): x > 0\}$

b) $\{(x,y,z): x + y + z = 1\}$

c) $\{(x,y,z): x + y + z = 0\}$

d) $\{(x,y,z): x^2 + y^2 + z^2 = 1\}$

SOLUTION

ANSWER (c)

Test $W = \{(x,y,z): x + y + z = 0\}$ with the three conditions:

1) $0 + 0 + 0 = 0$, so $(0,0,0) \in W$.

2) If $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$, then $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$: closed under addition.

3) If $x + y + z = 0$ and $k \in \mathbb{R}$, then $kx + ky + kz = k \cdot 0 = 0$: closed under scalar multiplication.

(a), (b) and (d) fail condition 1: $(0,0,0)$ does not satisfy $x > 0$, nor $x + y + z = 1$, nor $x^2 + y^2 + z^2 = 1$. A subspace must contain the zero vector.

SEMESTER 472 · QUESTION 1 (vii) · 1 MARK

What “linearly dependent” means

(vii) If a subset $S \neq \{0\}$ of a vector space V is linearly dependent, then:

a) The set S must span the vector space V.

b) No vector in S can be written as a linear combination of other vectors in S.

c) At least one vector in S is a linear combination of other vectors in S.

d) One vector in S must be 0.

SOLUTION

ANSWER (c)

A set of two or more vectors is linearly dependent exactly when at least one of its vectors is a linear combination of the others. (A one-vector set is dependent only if it is $\{0\}$, which is excluded.)

(a), (d) fail: $\{(1,0,0), (2,0,0)\}$ is dependent, does not span $\mathbb{R}^3$, and does not contain 0.

(b) describes a linearly independent set.

> A set that contains 0 is always dependent; a dependent set need not contain 0.

Transition matrices are invertible

(viii) Any transition matrix P must satisfy that:

a) P is diagonal **b)** $\det(P) = 0$ **c)** P is invertible **d)** P is symmetric

SOLUTION **ANSWER (c)**

${}_C P_B$ turns B-coordinates into C-coordinates and ${}_B P_C$ turns them back. Doing both changes nothing:

$$ {}_B P_C \; {}_C P_B = I, \qquad \text{so} \qquad {}_B P_C = ({}_C P_B)^{-1} $$

(b) $\det(P) = 0$ would make P singular, which never happens. (a), (d) can fail: the transition matrix in Question 3(a)(ii) of this same paper is neither diagonal nor symmetric.

Independent rows give the rank

(ix) If A is a 3×6 matrix with linearly independent rows, then:

a) $\operatorname{rank}(A)=6$ **b)** $\operatorname{rank}(A)=3$ **c)** $\operatorname{rank}(A)=4$ **d)** $\operatorname{rank}(A)=5$

SOLUTION

ANSWER (b)

The 3 rows span $\operatorname{row}(A)$ and are linearly independent, so they form a basis of $\operatorname{row}(A)$:

$$\operatorname{rank}(A)=\operatorname{dimrow}(A)=3$$

In any case $\operatorname{rank}(A)\le\min(3,6)=3$, so (a), (c) and (d) are impossible for a 3×6 matrix.

SEMESTER 472 · QUESTION 1 (x) · 1 MARK

Rank–nullity counts columns

(x) If a matrix A is of size 4×6 such that $\dim\big(\mathrm{col}(A)\big) = 3$ then $\dim\big(N(A)\big)$ is equal to:

a) ≤ 2 **b)** 1 **c)** 3 **d)** 0

SOLUTION **ANSWER (c)**

$\dim\big(\mathrm{col}(A)\big) = \mathrm{rank}(A) = 3$, and A has 6 columns. By the rank–nullity theorem:

$$\dim\big(N(A)\big) = \mathrm{nullity}(A) = 6 - \mathrm{rank}(A) = 6 - 3 = 3$$

Trap: using the 4 rows gives $4 - 3 = 1$, which is option (b). The theorem is $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$, the number of columns.

SEMESTER 472 · QUESTION 2 · 2 + 2 + 3 MARKS

Spanning sets and bases

(a) Give an example of a linearly dependent subset of the vector space $\mathbb{R}^3$ which spans $\mathbb{R}^3$. (2 marks)

(b) Let $\{u_1, u_2, u_3\}$ be a basis of a vector space V. Determine whether the set $\{2u_1,\ u_1 - 2u_2,\ 2u_1 + u_3\}$ is a basis of V or not. (2 marks)

(c) Let $A = \{v_1 = (1,1,1),\ v_2 = (2,1,1),\ v_3 = (1,0,0)\}$. Find a basis B of $\text{span}(A)$, and then find a basis C of $\mathbb{R}^3$ such that $B \subset C$. (3 marks)

A spanning set that is dependent

(a) Give an example of a linearly dependent subset of the vector space $\mathbb{R}^3$ which spans $\mathbb{R}^3$.

SOLUTION

Start from a basis of $\mathbb{R}^3$ and add one more vector. The key's example:

$$\{(1,1,1),\ (1,0,0),\ (0,1,0),\ (0,0,1)\}$$

- **It spans $\mathbb{R}^3$:** it contains the standard basis, and $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$.
- **It is dependent:** four vectors in $\mathbb{R}^3$; explicitly, $(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$.

Adding any vector to a basis of $\mathbb{R}^3$ gives a dependent spanning set: more than 3 vectors in $\mathbb{R}^3$ are always dependent.

Is $\{2u_1,\ u_1 - 2u_2,\ 2u_1 + u_3\}$ a basis?

(b) Let $\{u_1, u_2, u_3\}$ be a basis of a vector space V. Determine whether the set $\{2u_1,\ u_1 - 2u_2,\ 2u_1 + u_3\}$ is a basis of V or not.

SOLUTION

Independence. Suppose $\alpha(2u_1) + \beta(u_1 - 2u_2) + \gamma(2u_1 + u_3) = 0$. Collect the terms:

$$(2\alpha + \beta + 2\gamma)u_1 - 2\beta u_2 + \gamma u_3 = 0$$

$\{u_1, u_2, u_3\}$ is linearly independent, so every coefficient is 0:

$$\gamma = 0, \qquad -2\beta = 0 \ \Rightarrow\ \beta = 0, \qquad 2\alpha + \beta + 2\gamma = 0 \ \Rightarrow\ \alpha = 0$$

Basis. Three linearly independent vectors in V, and $\dim V = 3$: they form a basis of V.

Shortcut: their coordinate vectors relative to $\{u_1, u_2, u_3\}$ are the columns of $\begin{bmatrix} 2 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, a triangular matrix with determinant $2(-2)(1) = -4 \neq 0$.

Reduce to a basis of $\text{span}(A)$, then extend

(c) Let $A = \{v_1 = (1,1,1),\ v_2 = (2,1,1),\ v_3 = (1,0,0)\}$. Find a basis B of $\text{span}(A)$, and then find a basis C of $\mathbb{R}^3$ such that $B \subset C$.

SOLUTION

Reduce. $v_1 = v_2 - v_3$, since $(2,1,1) - (1,0,0) = (1,1,1)$. So $\text{span}(A) = \text{span}\{v_2, v_3\}$, and $v_2,\ v_3$ are not multiples of each other: $B = \{v_2, v_3\}$ is a basis of $\text{span}(A)$.

Extend. Every vector of $\text{span}(B)$ is $a(2,1,1) + b(1,0,0) = (2a + b,\ a,\ a)$: its second and third entries are equal. So $(0,1,0) \notin \text{span}(B)$, and adding it keeps the set independent:

$$C = \{v_2,\ v_3,\ (0,1,0)\}, \qquad \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \neq 0$$

$\text{span}(A) = \{(x,y,z) \in \mathbb{R}^3 : y = z\}$, a plane through the origin. Any vector with $y \neq z$ extends B to a basis of $\mathbb{R}^3$.

Coordinates, a transition matrix, rank and nullity

(a) Consider the two ordered bases $B = \{u_1 = (0,1,1),\ u_2 = (1,1,1),\ u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1),\ v_2 = (1,1,0),\ v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(i) Find the coordinate vector $[v]_B$ of the vector v with respect to the ordered basis B. (2 marks)

(ii) Construct the transition matrix $P_{B\to C}$ from the basis B to the basis C, and then use it to find the coordinate vector $[v]_C$ of the vector v with respect to the ordered basis C. (3 marks)

(b) Find the rank and nullity of the matrix $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. (3 marks)

The coordinate vector $[v]_B$

(a) Consider the two ordered bases $B = \{u_1 = (0,1,1),\ u_2 = (1,1,1),\ u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1),\ v_2 = (1,1,0),\ v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(i) Find the coordinate vector $[v]_B$ of the vector v with respect to the ordered basis B.

SOLUTION

$$v = au_1 + bu_2 + cu_3: \quad \begin{bmatrix}1\\2\\1\end{bmatrix} = a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}b+c\\a+b\\a+b+c\end{bmatrix}$$

$$\begin{aligned} b + c &= 1 \\ a + b &= 2 \\ a + b + c &= 1 \end{aligned}$$

Subtract the second equation from the third: $c = -1$. Then $b = 1 - c = 2$ and $a = 2 - b = 0$.

$[v]_B = (0,2,-1)^T$. Check: $0u_1 + 2u_2 - u_3 = (2,2,2) - (1,0,1) = (1,2,1) = v$.

The transition matrix ${}_C P_B$, then $[v]_C$

(a) Consider the two ordered bases $B = \{u_1 = (0,1,1),\ u_2 = (1,1,1),\ u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1),\ v_2 = (1,1,0),\ v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(ii) Construct the transition matrix $P_{B\to C}$ from the basis B to the basis C, and then use it to find the coordinate vector $[v]_C$ of the vector v with respect to the ordered basis C.

SOLUTION

Column j of ${}_C P_B$ is $[u_j]_C$. Reduce $[\,v_1\ v_2\ v_3 \mid u_1\ u_2\ u_3\,]$ to $[\,I \mid {}_C P_B\,]$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{I_{1,2},\ A^{(1)}_{1,3},\ A^{(-1)}_{2,1},\ A^{(-1)}_{2,3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{M^{(1/2)}_3,\ A^{(-1)}_{3,1},\ A^{(1)}_{3,2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & -1 \\ 0 & 1 & 0 & 1 & 3/2 & 1 \\ 0 & 0 & 1 & 1 & 1/2 & 0 \end{array}\right]$$

$$[v]_C = {}_C P_B\,[v]_B = \begin{bmatrix} 0 & -1/2 & -1 \\ 1 & 3/2 & 1 \\ 1 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Check: $0v_1 + 2v_2 + v_3 = (2,2,0) + (-1,0,1) = (1,2,1) = v$.

Rank and nullity of a 4×4 matrix

(b) Find the rank and nullity of the matrix $\begin{bmatrix}1&2&1&1\\2&1&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}$.

SOLUTION

$$\begin{bmatrix}1&2&1&1\\2&1&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix} \xrightarrow{A_{1,2}^{(-2)},\, A_{1,3}^{(-1)},\, A_{1,4}^{(-1)}} \begin{bmatrix}1&2&1&1\\0&-3&-1&-1\\0&-1&1&0\\0&-1&0&1\end{bmatrix} \xrightarrow{I_{2,3},\, M_2^{(-1)}} \begin{bmatrix}1&2&1&1\\0&1&-1&0\\0&-3&-1&-1\\0&-1&0&1\end{bmatrix}$$

$$\xrightarrow{A_{2,3}^{(3)},\, A_{2,4}^{(1)}} \begin{bmatrix}1&2&1&1\\0&1&-1&0\\0&0&-4&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{I_{3,4},\, M_3^{(-1)},\, A_{3,4}^{(4)},\, M_4^{(-1/5)}} \begin{bmatrix}1&2&1&1\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

Four pivots, one in every column: $\text{rank} = 4$ and $\text{nullity} = 4 - 4 = 0$.

A square matrix with a pivot in every column is invertible: here $\det = -5 \neq 0$.