MIDTERM 1 · STATS AND SOLUTIONS

Midterm Exam 1 Statistics & Solutions

Semester 481 · Sunday 11 October 2026

MATH 244 — Linear Algebra
Sections 1.1 – 1.7 and 2.1 – 2.3
King Saud University · Department of Mathematics

QUESTION 1 (1) · 1 POINT

Products of upper triangular matrices

(1) Any upper triangular matrix A satisfies

a) A^2 is upper triangular

b) $A^2 - I$ is invertible

c) AA^T is upper triangular

d) $A - A^T$ is diagonal

SOLUTION

ANSWER (a)

A product of upper triangular matrices is upper triangular, so $A^2 = AA$ is upper triangular.

One matrix rules out the other three choices: take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$$A^2 - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

(b) is singular

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(c) is not upper triangular

$$A - A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(d) is not diagonal

Upper triangular matrices stay upper triangular under sums, scalar multiples and products, but not under transposition: A^T is lower triangular.

QUESTION 1 (2) · 1 POINT

When is a diagonal matrix invertible?

(2) Any real diagonal matrix A satisfies

a) A is invertible

b) $A - I$ is invertible

c) $A + I$ is invertible

d) $A^2 + I$ is invertible

SOLUTION **ANSWER (d)**

A diagonal matrix is invertible exactly when all its diagonal entries are nonzero. If $A = \mathrm{diag}(d_1, \ldots, d_n)$ with real d_i, then

$$A^2 + I = \mathrm{diag}\big(d_1^2 + 1,\ \ldots,\ d_n^2 + 1\big), \qquad d_i^2 + 1 \geq 1 > 0$$

so $A^2 + I$ is always invertible. Simple diagonal matrices rule out the rest:

$A = O$	$A = I$	$A = -I$
(a) A is singular	(b) $A - I = O$	(c) $A + I = O$

The word real matters: a complex entry $d = i$ would give $d^2 + 1 = 0$.

QUESTION 1 (3) · 1 POINT

A 2×2 system

(3) The solution of the linear system $\begin{cases} -3x + y = -1 \\ 5x - 2y = 1 \end{cases}$ is:

a) $(3,3)$ **b)** $(-2,1)$ **c)** $(1,2)$ **d)** $(1,-1)$

SOLUTION **ANSWER (c)**

In a multiple-choice question it is quickest to substitute each option into the first equation, $-3x + y = -1$:

	a) $(3,3)$	**b)** $(-2,1)$	**c)** $(1,2)$	**d)** $(1,-1)$
$-3x + y$	-6	7	-1	-4

Only (c) gives -1, and it satisfies the second equation too: $5(1) - 2(2) = 1$.

Directly: the coefficient matrix has determinant $(-3)(-2) - (1)(5) = 1$, so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

QUESTION 1 (4) · 1 POINT

An inverse hidden in $A^2 + A = I$

(4) If A is a square matrix such that $A^2 + A = I$, then:

a) A is singular **b)** $A^{-1} = I - A$ **c)** $A^2 = I$ **d)** $A^{-1} = A + I$

SOLUTION **ANSWER (d)**

Factor out A, on the left and on the right:

$$A(A + I) = A^2 + A = I \qquad \text{and} \qquad (A + I)A = A^2 + A = I$$

So A is invertible, with $A^{-1} = A + I$. In particular (a) is false.

(b): $A(I - A) = A - A^2 = 2A - I$, which is I only when $A = I$, and $A = I$ does not satisfy $A^2 + A = I$. (c): $A^2 = I - A$, so $A^2 = I$ would force $A = 0$, which does not satisfy it either.

QUESTION 1 (5) · 1 POINT

Reading solutions from a reduced matrix

(5) The set of solutions of $Ax = 0$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, is:

a) $\{(0, t, 0) : t \in \mathbb{R}\}$ **b)** $\{(t, 0, 0) : t \in \mathbb{R}\}$ **c)** $\{(0, 0, t) : t \in \mathbb{R}\}$ **d)** $\{(t, t, 0) : t \in \mathbb{R}\}$

SOLUTION **ANSWER (a)**

A is already in reduced row echelon form. Its two nonzero rows say

$$x_1 = 0, \qquad x_3 = 0$$

Column 2 has no pivot, so x_2 is free: put $x_2 = t$. The solution set is $\{(0, t, 0) : t \in \mathbb{R}\}$.

Check the other options with $t = 1$: $(1,0,0)$ and $(1,1,0)$ break $x_1 = 0$, and $(0,0,1)$ breaks $x_3 = 0$.

QUESTION 1 (6) · 1 POINT

When is this matrix singular?

(6) The matrix $\begin{bmatrix} 3 & 6 \\ 1 & 2x \end{bmatrix}$ is singular if:

a) $x = 0$ **b)** $x = 1$ **c)** $x = 3$ **d)** $x = 2$

SOLUTION **ANSWER (b)**

A square matrix is singular exactly when its determinant is 0:

$$\begin{vmatrix} 3 & 6 \\ 1 & 2x \end{vmatrix} = 3(2x) - 6(1) = 6x - 6 = 0 \Leftrightarrow x = 1$$

For $x = 1$ the rows $(3, 6)$ and $(1, 2)$ are proportional: row 1 is 3 times row 2.

Every other option gives a nonzero determinant: $x = 0$ gives -6, $x = 3$ gives 12, and $x = 2$ gives 6.

QUESTION 1 (7) · 1 POINT

Which elementary matrix produced EA?

(7) Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 4 & 0 \\ 1 & -1 & -2 \end{bmatrix}$ and $EA = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 4 & 0 \\ -1 & 2 & -3 \end{bmatrix}$. Then the elementary matrix E is equal to:

a) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

SOLUTION **ANSWER (b)**

Rows 1 and 2 of EA are those of A, and row 3 is $(-1,2,-3) = (1,-1,-2) - (2,-3,1)$, that is, $R_3 - R_1$. So E performs $A_{1,3}^{(-1)}$; apply it to I_3:

$$I_3 \xrightarrow{A_{1,3}^{(-1)}} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Options (a), (c) and (d) change row 1, row 2, and rows 1 and 3, which all stayed the same.

QUESTION 1 (8) · 1 POINT

Both invertible and symmetric

(8) Let A be an invertible matrix. Which of the following matrices must be **both** invertible and symmetric?

a) AA^T **b)** $A + A^T$ **c)** $A - A^T$ **d)** A^{-1}

SOLUTION **ANSWER (a)**

AA^T is symmetric and invertible for every invertible A:

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T, \qquad \left|AA^T\right| = |A|\left|A^T\right| = |A|^2 \neq 0$$

One invertible matrix rules out the rest: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, with $|A| = 1$.

$A + A^T = O$

(b) is not invertible

$A - A^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(c) is not symmetric

$A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) is not symmetric

QUESTION 1 (9) · 1 POINT

When $Ax = b$ has no solution

(9) Let A be a 3×3 matrix. If the system $Ax = \begin{bmatrix}1\\2\\3\end{bmatrix}$ has no solution, then:

a) $Ax = 0$ has no solution

b) $Ax = 0$ has infinitely many solutions

c) $Ax = 0$ has a unique solution

d) $A^T x = \begin{bmatrix}1\\2\\3\end{bmatrix}$ has a unique solution

SOLUTION **ANSWER (b)**

If A were invertible, $x = A^{-1}b$ would be a solution. So A is singular, and then $Ax = 0$ has a solution $x_0 \neq 0$; every multiple tx_0 is a solution too, so there are infinitely many.

(a) never true: $x = 0$ always solves $Ax = 0$

(c) a unique solution would make A invertible

(d) $|A^T| = |A| = 0$, so A^T is singular too

Example: $A = \mathrm{diag}(1,1,0)$. Then $Ax = \begin{bmatrix}1\\2\\3\end{bmatrix}$ would need $0 = 3$, while $Ax = 0$ is solved by every $t(0,0,1)$.

QUESTION 1 (10) · 1 POINT

The determinant of a product

(10) Let A and B be 3×3 matrices with $|A| = 2$ and $|B| = -3$. Then $\left|2B^T A^2 B^{-1}\right|$ is equal to:

a) 8 **b)** 16 **c)** 32 **d)** -32

SOLUTION **ANSWER (c)**

The matrix inside is 3×3, so the factor 2 contributes 2^3. Then use $\left|B^T\right| = |B|$ and $\left|B^{-1}\right| = \frac{1}{|B|}$:

$$\left|2B^T A^2 B^{-1}\right| = 2^3 \left|B^T\right| |A|^2 \left|B^{-1}\right| = 8 \cdot (-3) \cdot 4 \cdot \left(-\frac{1}{3}\right) = 32$$

Traps: writing $|2M| = 2|M|$ gives 8, option (a); a sign slip in $\left|B^{-1}\right| = -\frac{1}{3}$ gives -32, option (d). The value of $|B|$ cancels out.

QUESTION 2 · 5 POINTS

One solution, none, or infinitely many?

Consider the following system of linear equations. Find the values of m such that the system has (i) a unique solution, (ii) no solution, (iii) infinitely many solutions.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 1 \\ 2x_1 - 3x_2 + (m+5)x_3 = -1 \\ x_1 + (m-2)x_2 + 3x_3 = -2 \end{cases}$$

SOLUTION

$$\begin{bmatrix} 1 & -2 & 3 & | & 1 \\ 2 & -3 & m+5 & | & -1 \\ 1 & m-2 & 3 & | & -2 \end{bmatrix} \xrightarrow{A_{1,2}^{(-2)},\, A_{1,3}^{(-1)}} \begin{bmatrix} 1 & -2 & 3 & | & 1 \\ 0 & 1 & m-1 & | & -3 \\ 0 & m & 0 & | & -3 \end{bmatrix} \xrightarrow{A_{2,3}^{(-m)}} \begin{bmatrix} 1 & -2 & 3 & | & 1 \\ 0 & 1 & m-1 & | & -3 \\ 0 & 0 & -m(m-1) & | & 3(m-1) \end{bmatrix}$$

	$m = 0$	$m = 1$	$m \neq 0,\ m \neq 1$
Last row	$0x_3 = -3$: impossible	$0x_3 = 0$: x_3 is free	$x_3 = -3/m$
Solutions	(ii) none	(iii) infinitely many	(i) exactly one

(i) $m \in \mathbb{R}\backslash\{0,1\}$ (ii) $m = 0$ (iii) $m = 1$

Check: the coefficient determinant is $-m(m-1)$, which is nonzero exactly when $m \neq 0,1$.

QUESTION 3 · 3 POINTS

A homogeneous system with two free variables

Solve the following homogeneous linear system:

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ -x_1 + x_3 - 3x_4 = 0 \\ x_1 - 2x_2 + 5x_3 - x_4 = 0 \\ -x_2 + 3x_3 - 2x_4 = 0 \end{cases}$$

SOLUTION

The right-hand side is zero and stays zero, so reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 0 & 1 & -3 \\ 1 & -2 & 5 & -1 \\ 0 & -1 & 3 & -2 \end{bmatrix} \xrightarrow{A_{1,2}^{(1)},\, A_{1,3}^{(-1)}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & -1 & 3 & -2 \\ 0 & -1 & 3 & -2 \end{bmatrix} \xrightarrow{A_{2,3}^{(-1)},\, A_{2,4}^{(-1)},\, M_2^{(-1)},\, A_{2,1}^{(1)}} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots in columns 1 and 2, so $x_3 = s$ and $x_4 = t$ are free: $x_1 = s - 3t$ and $x_2 = 3s - 2t$.

Solution set: $\{(s - 3t,\ 3s - 2t,\ s,\ t) : s, t \in \mathbb{R}\} = \{s(1,3,1,0) + t(-3,-2,0,1) : s, t \in \mathbb{R}\}$.

QUESTION 4 · 3 POINTS

Working from A^{-1}

If $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$, find:

(i) $\left(A^T\right)^{-1}$
(ii) $\text{adj}(A)$

Neither part requires computing A itself.

QUESTION 4 (i)

The inverse of the transpose

If $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$, find:

(i) $\left(A^T\right)^{-1}$

SOLUTION

Transposing $AA^{-1} = I$ gives $\left(A^{-1}\right)^T A^T = I^T = I$, so A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$:

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

No inversion is needed: transpose the given matrix.

QUESTION 4 (ii)

The adjoint without computing A

If $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$, find:

(ii) $\text{adj}(A)$

SOLUTION

Since $A^{-1} = \frac{1}{|A|}\text{adj}(A)$, we have $\text{adj}(A) = |A|\,A^{-1}$ with $|A| = \frac{1}{|A^{-1}|}$:

$$\left|A^{-1}\right| = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0 + 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 3 - 1 = 2, \qquad |A| = \frac{1}{2}$$

$$\text{adj}(A) = \frac{1}{2}A^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 1/2 & -1/2 \\ 3/2 & 1/2 & 1 \end{bmatrix}$$

QUESTION 5 · 4 POINTS

Cramer's rule with a parameter

Consider the following linear system:

$$\begin{cases} \alpha x - y - z = -1 \\ -x + \alpha y + z = 1 \\ x + 2y + z = 1 \end{cases}$$

(i) Find the values of α for which the above system can be solved using Cramer's rule.

(ii) Assuming the system can be solved by Cramer's rule, find the value of y.

QUESTION 5 (i)

When does Cramer's rule apply?

Consider the following linear system:

(i) Find the values of α for which the above system can be solved using Cramer's rule.

$$\begin{cases} \alpha x - y - z = -1 \\ -x + \alpha y + z = 1 \\ x + 2y + z = 1 \end{cases}$$

SOLUTION

Cramer's rule needs the coefficient determinant to be nonzero. Expand along row 1:

$$D = \begin{vmatrix} \alpha & -1 & -1 \\ -1 & \alpha & 1 \\ 1 & 2 & 1 \end{vmatrix} = \alpha(\alpha - 2) - (-1)(-2) + (-1)(-2 - \alpha) = \alpha^2 - \alpha = \alpha(\alpha - 1)$$

Cramer's rule applies exactly when $\alpha(\alpha - 1) \neq 0$, that is, $\alpha \neq 0$ and $\alpha \neq 1$.

QUESTION 5 (ii)

Finding y by Cramer's rule

Consider the following linear system:

(ii) Assuming the system can be solved by Cramer's rule, find the value of y.

$$\begin{cases} \alpha x - y - z = -1 \\ -x + \alpha y + z = 1 \\ x + 2y + z = 1 \end{cases}$$

SOLUTION

Replace column 2 of the coefficient matrix by the right-hand side $(-1,1,1)$. The result has two equal columns, so its determinant is 0:

$$D_y = \begin{vmatrix} \alpha & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \alpha(1-1) - (-1)(-1-1) + (-1)(-1-1) = 0 - 2 + 2 = 0$$

$$y = \frac{D_y}{D} = \frac{0}{\alpha(\alpha-1)} = 0 \qquad (\alpha \neq 0,\ 1)$$

Why: the right-hand side equals column 3 of the coefficient matrix, so $(x, y, z) = (0,0,1)$ solves the system; for $\alpha \neq 0,1$ it is the only solution.