MIDTERM 1 REVIEW

Previous Midterm 1 Exams

Semesters 471 and 472 · Sections 1.1 – 1.7 and 2.1 – 2.3

MATH 244 — Linear Algebra
Midterm 1 · Sunday 11 October 2026
King Saud University · Department of Mathematics

SEMESTER 471 · MIDTERM 1

Midterm Exam 1, Semester 471

25 marks · 1½ hours · three questions. Work each question on paper first, then go through its parts one slide at a time.

1. Question 1 — true or false: five statements (5 × 1 mark)
2. Question 2 — a 4×4 determinant; working from A^{-1} (5 + 5 marks)
3. Question 3 — Gauss–Jordan elimination; a homogeneous system (5 + 5 marks)

Determine whether each statement is true or false

(i) If A and B are diagonal matrices of the same size, then $AB = BA$.

(ii) If A and B are $n \times n$ matrices and A is singular, then AB is also singular.

(iii) If a matrix A satisfies $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.

(iv) Any square matrix A satisfies $\mathrm{RREF}(A) = I$ if and only if A is invertible.

(v) If the linear system $AX = B$ has solutions $u = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, then $w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ is another solution of the same system.

One mark each. A TRUE statement needs a reason that works for every matrix; a FALSE one needs a single counterexample.

Do diagonal matrices commute?

(i) If A and B are diagonal matrices of the same size, then $AB = BA$.

SOLUTION

TRUE

Write $A = \operatorname{diag}(a_{11}, \dots, a_{nn})$ and $B = \operatorname{diag}(b_{11}, \dots, b_{nn})$. A product of diagonal matrices is diagonal, and its diagonal entries are the products of the matching entries:

$$AB = \begin{bmatrix} a_{11}b_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn}b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn}a_{nn} \end{bmatrix} = BA$$

The middle equality is only $a_{ii}b_{ii} = b_{ii}a_{ii}$: real numbers commute.

Diagonal matrices of the same size always commute. Most pairs of matrices do not — Semester 472, Question 2(a) is built on that.

A singular factor makes the product singular

(ii) If A and B are $n \times n$ matrices and A is singular, then AB is also singular.

SOLUTION

TRUE

A is singular, so $|A| = 0$. The determinant of a product is the product of the determinants:

$$|AB| = |A|\,|B| = 0 \cdot |B| = 0$$

A square matrix with determinant 0 is not invertible, so AB is singular.

The same one-line argument shows that BA and every power A^k are singular as well.

The identity has many square roots

> **(iii)** If a matrix A satisfies $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.

SOLUTION **FALSE**

One counterexample settles it. The 2×2 matrix that swaps the two rows works:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

yet $A \neq I_2$ and $A \neq -I_2$.

> Why the number rule fails: $A^2 = I$ gives $(A - I)(A + I) = O$, and two nonzero matrices can multiply to O. Here $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Reduced row echelon form and invertibility

(iv) Any square matrix A satisfies $\mathrm{RREF}(A) = I$ if and only if A is invertible.

SOLUTION

TRUE

- Each elementary row operation multiplies the determinant by a nonzero number: -1 for a swap, k for scaling a row by $k \neq 0$, 1 for a replacement. So $|\mathrm{RREF}(A)| = c\,|A|$ with $c \neq 0$.
- A square matrix in reduced row echelon form is either I, with determinant 1, or it has a row of zeros, with determinant 0.

$$\mathrm{RREF}(A) = I \Leftrightarrow |\mathrm{RREF}(A)| \neq 0 \Leftrightarrow |A| \neq 0 \Leftrightarrow A \text{ is invertible}$$

This is one line of the list of equivalent statements: A invertible $\Leftrightarrow$ $AX = O$ has only the trivial solution $\Leftrightarrow$ $\mathrm{RREF}(A) = I_n \Leftrightarrow |A| \neq 0$.

The midpoint of two solutions

(v) If the linear system $AX = B$ has solutions $u = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, then $w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ is another solution of the same system.

SOLUTION **TRUE**

The vector w is the midpoint of u and v: $w = \frac{1}{2}(u + v)$. Using $Au = B$ and $Av = B$:

$$Aw = A\left(\frac{1}{2}u + \frac{1}{2}v\right) = \frac{1}{2}Au + \frac{1}{2}Av = \frac{1}{2}B + \frac{1}{2}B = B$$

So w is another solution of $AX = B$.

More generally, $su + tv$ is a solution whenever $s + t = 1$. So a system with two different solutions has infinitely many.

Determinants and the inverse

(a) Let $A = \begin{bmatrix} a & a & a & a \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{bmatrix}$, where $a \neq 0$ and $b \neq a$. Show that $|A| = a(a-b)^3$ and deduce that the matrix A is invertible. (5 marks)

(b) Let A be the matrix with $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Find $\left(A^T\right)^{-1}$, $|2A|$, $\text{adj}(A)$ and $\text{RREF}(A)$. Also find the solution of the homogeneous linear system $AX = O$. (5 marks)

A patterned 4×4 determinant

(a) Let $A = \begin{bmatrix} a & a & a & a \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{bmatrix}$, where $a \neq 0$ and $b \neq a$. Show that $|A| = a(a-b)^3$ and deduce that the matrix A is invertible.

SOLUTION

Take a out of row 1, then subtract b times row 1 from rows 2, 3 and 4 (this does not change the determinant):

$$|A| = a \begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{vmatrix} \overset{A_{1,2}^{(-b)}, A_{1,3}^{(-b)}, A_{1,4}^{(-b)}}{=} a \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-b & a-b & a-b \\ 0 & 0 & a-b & a-b \\ 0 & 0 & 0 & a-b \end{vmatrix} = a \cdot 1 \cdot (a-b)^3$$

The last determinant is triangular: multiply its diagonal. Since $a \neq 0$ and $a - b \neq 0$, we get $|A| = a(a-b)^3 \neq 0$, so A is invertible.

Five answers that follow from A^{-1}

(b) Let A be the matrix with $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Find $\left(A^T\right)^{-1}$, $|2A|$, $\text{adj}(A)$ and $\text{RREF}(A)$. Also find the solution of the homogeneous linear system $AX = O$.

SOLUTION

1. $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 3 & 8 \end{bmatrix}$

2. $\left|A^{-1}\right| = 1(40-0) - 2(16-3) + 3(0-5) = -1$, so $|A| = \frac{1}{|A^{-1}|} = -1$ and $|2A| = 2^3|A| = -8$.

3. $\text{adj}(A) = |A|\,A^{-1} = -A^{-1} = \begin{bmatrix} -1 & -2 & -3 \\ -2 & -5 & -3 \\ -1 & 0 & -8 \end{bmatrix}$

4. $|A| \neq 0$, so A is invertible and $\text{RREF}(A) = I_3$.

5. $X = A^{-1}O = O$: only the trivial solution.

Rules used: $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$, $|kA| = k^n|A|$, $\left|A^{-1}\right| = \frac{1}{|A|}$, $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

SEMESTER 471 · QUESTION 3 · 5 + 5 MARKS

Two linear systems

(a) Solve the following system of equations by the Gauss–Jordan elimination method: (5 marks)

$$\begin{aligned} w + x + y + z &= 6 \\ w + y + z &= 4 \\ w + y &= 2 \end{aligned}$$

(b) Under what condition on α and β does the following system of equations have no nontrivial solution? (5 marks)

$$\begin{aligned} x + 2z &= 0 \\ \alpha x + 8y + 3z &= 0 \\ \beta y + 5z &= 0 \end{aligned}$$

Gauss–Jordan elimination with a free variable

(a) Solve the following system of equations by the Gauss–Jordan elimination method:

$$\begin{aligned} w &+ x + y + z = 6 \\ w &\quad + y + z = 4 \\ w &\quad + y \quad\;\; = 2 \end{aligned}$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 6 \\ 1 & 0 & 1 & 1 & | & 4 \\ 1 & 0 & 1 & 0 & | & 2 \end{bmatrix} \xrightarrow{A_{1,2}^{(-1)},\, A_{1,3}^{(-1)}} \begin{bmatrix} 1 & 1 & 1 & 1 & | & 6 \\ 0 & -1 & 0 & 0 & | & -2 \\ 0 & -1 & 0 & -1 & | & -4 \end{bmatrix} \xrightarrow{M_2^{(-1)},\, A_{2,1}^{(-1)},\, A_{2,3}^{(1)}} \begin{bmatrix} 1 & 0 & 1 & 1 & | & 4 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 0 & -1 & | & -2 \end{bmatrix}$$

$$\xrightarrow{M_3^{(-1)},\, A_{3,1}^{(-1)}} \begin{bmatrix} 1 & 0 & 1 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \quad \text{(RREF)}$$

The RREF reads $w + y = 2$, $x = 2$, $z = 2$. Column 3 has no pivot, so y is free: put $y = t$.

Solution set: $\{(w, x, y, z) = (2 - t,\ 2,\ t,\ 2) : t \in \mathbb{R}\}$ — infinitely many solutions.

When is the trivial solution the only one?

(b) Under what condition on α and β does the following system of equations have no nontrivial solution?

$$\begin{aligned} x &+ 2z = 0 \\ \alpha x + 8y &+ 3z = 0 \\ \beta y &+ 5z = 0 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ \alpha & 8 & 3 & 0 \\ 0 & \beta & 5 & 0 \end{array}\right] \xrightarrow{A_{1,2}^{(-\alpha)}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 8 & 3-2\alpha & 0 \\ 0 & \beta & 5 & 0 \end{array}\right] \xrightarrow{M_2^{(1/8)},\, A_{2,3}^{(-\beta)},\, M_3^{(8)}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{3-2\alpha}{8} & 0 \\ 0 & 0 & 40-(3-2\alpha)\beta & 0 \end{array}\right]$$

Only the trivial solution ⇔ a pivot in every column ⇔ the (3,3) entry is not zero.

> No nontrivial solution exactly when $40-(3-2\alpha)\beta \neq 0$, that is, $(3-2\alpha)\beta \neq 40$.

Check: the coefficient determinant is $40-3\beta+2\alpha\beta = 40-(3-2\alpha)\beta$, and a square homogeneous system has only the trivial solution exactly when its determinant is not zero.

SEMESTER 472 · MIDTERM 1

Midterm Exam 1, Semester 472

25 marks · 1½ hours · calculators not allowed. Work each question on paper first, then go through its parts one slide at a time.

1. Question 1 — five multiple-choice items (5 × 1 mark)
2. Question 2 — matrix algebra, right inverses and a 4×4 determinant (4 + 4 + 2 marks)
3. Question 3 — a system with a parameter; the inverse by the adjoint (6 + 4 marks)

Which of the given choices is correct?

(i) If A is a square matrix, $A^2 = 0$ and I is the identity matrix, then $(2A + I)^{-1}$ is equal to:

a) $I - A$ **b)** $I + A$ **c)** $I - 2A$ **d)** $2I - A$

(ii) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then the unique solution of the equation $A^{-1} = \alpha A + \beta I$ is:

a) $\alpha = \frac{1}{10}, \beta = -\frac{1}{5}$ **b)** $\alpha = \frac{1}{10}, \beta = -\frac{3}{10}$ **c)** $\alpha = -\frac{1}{10}, \beta = \frac{1}{2}$ **d)** $\alpha = \frac{1}{10}, \beta = -\frac{1}{2}$

(iii) Which of the following matrices is not an elementary matrix?

a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(iv) If $A\,\mathrm{adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the determinant of the matrix A satisfies:

a) $|A| > 3$ **b)** $|A| < 5$ **c)** $|A| = 9$ **d)** $|A| = 27$

(v) If I is a row echelon form of the matrix A, then the linear system $AX = B$ has:

a) infinitely many solutions **b)** no solutions **c)** a unique solution **d)** more than one solution

SEMESTER 472 · QUESTION 1 (i) · 1 MARK

The inverse of $2A + I$ when $A^2 = 0$

(i) If A is a square matrix, $A^2 = 0$ and I is the identity matrix, then $(2A + I)^{-1}$ is equal to:

a) $I - A$ **b)** $I + A$ **c)** $I - 2A$ **d)** $2I - A$

SOLUTION **ANSWER (c)**

For square matrices one product is enough: if $MN = I$ then $N = M^{-1}$. Multiply $2A + I$ by option (c):

$$(2A + I)(I - 2A) = 2A - 4A^2 + I - 2A = I - 4A^2 = I$$

because $A^2 = 0$. Hence $(2A + I)^{-1} = I - 2A$.

The other options fail: $(2A + I)(I - A) = I + A$, $(2A + I)(I + A) = I + 3A$ and $(2A + I)(2I - A) = 2I + 3A$, none of which is I unless $A = 0$.

Writing A^{-1} as $\alpha A + \beta I$

(ii) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then the unique solution of the equation $A^{-1} = \alpha A + \beta I$ is:

a) $\alpha = \frac{1}{10},\ \beta = -\frac{1}{5}$ **b)** $\alpha = \frac{1}{10},\ \beta = -\frac{3}{10}$ **c)** $\alpha = -\frac{1}{10},\ \beta = \frac{1}{2}$ **d)** $\alpha = \frac{1}{10},\ \beta = -\frac{1}{2}$

SOLUTION **ANSWER (c)**

$|A| = 4 - (-6) = 10$, so $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$. Compare entries with $\alpha A + \beta I = \begin{bmatrix} \alpha + \beta & 2\alpha \\ -3\alpha & 4\alpha + \beta \end{bmatrix}$:

$$2\alpha = -\frac{2}{10} \Rightarrow \alpha = -\frac{1}{10}, \qquad \alpha + \beta = \frac{4}{10} \Rightarrow \beta = \frac{1}{2}$$

The two remaining entries agree: $-3\alpha = \frac{3}{10}$ and $4\alpha + \beta = \frac{1}{10}$.

A quicker route: $A^2 = \begin{bmatrix} -5 & 10 \\ -15 & 10 \end{bmatrix} = 5A - 10I$, so $A(5I - A) = 10I$ and $A^{-1} = -\frac{1}{10}A + \frac{1}{2}I$.

Which matrix is not elementary?

(iii) Which of the following matrices is not an elementary matrix?

a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

SOLUTION **ANSWER (b)**

An elementary matrix comes from I by exactly ONE elementary row operation. Find the operation for each option:

a) $M_1^{(2)}$ on I_2	**b)** $A_{1,2}^{(1)}$ and $A_{1,3}^{(1)}$ on I_3	**c)** $A_{2,1}^{(1)}$ on I_3	**d)** $A_{1,3}^{(2)}$ on I_3
elementary	two operations: NOT elementary	elementary	elementary

Notation: $I_{i,j}$ swaps rows i and j; $M_i^{(k)}$ multiplies row i by $k \neq 0$; $A_{i,j}^{(k)}$ adds k times row i to row j.

Reading $|A|$ from $A\,\mathrm{adj}(A)$

(iv) If $A\,\mathrm{adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the determinant of the matrix A satisfies:

a) $|A| > 3$ **b)** $|A| < 5$ **c)** $|A| = 9$ **d)** $|A| = 27$

SOLUTION **ANSWER (b)**

Every square matrix satisfies $A\,\mathrm{adj}(A) = |A|\,I$. Here

$$A\,\mathrm{adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I \quad \Rightarrow \quad |A| = 3$$

With $|A| = 3$: (a) $3 > 3$ is false, (b) $3 < 5$ is TRUE, (c) and (d) are false.

Traps: $|\mathrm{adj}(A)| = |A|^2 = 9$ gives option (c), and $|A\,\mathrm{adj}(A)| = |3I| = 27$ gives option (d). Neither number is $|A|$.

When I is an echelon form of A

(v) If I is a row echelon form of the matrix A, then the linear system $AX = B$ has:

a) infinitely many solutions

b) no solutions

c) a unique solution

d) more than one solution

SOLUTION

ANSWER (c)

If I is a row echelon form of A, then A is square and row equivalent to I, so A is invertible.

The same row operations take $[A \mid B]$ to $[I \mid C]$: every variable has a pivot and the system is consistent, so

$$X = A^{-1}B \text{ is the only solution.}$$

Options (a) and (d) need a free variable, that is, a column without a pivot; option (b) needs a row $[\,0 \;\cdots\; 0 \mid c\,]$ with $c \neq 0$. Neither can happen when A reduces to I.

Matrix algebra and a determinant

(a) Let A and B be two square matrices of the same size. Then: (2 + 2 marks)

(i) Show that $(A+B)^2 = A^2 + 2AB + B^2$ if $(A+B)(A-B) = A^2 - B^2$.

(ii) Give an example of matrices A and B for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find all matrices B such that $AB = I$. (4 marks)

(c) Show that $\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = (a+b)(a-b)^3$. (2 marks)

SEMESTER 472 · QUESTION 2 (a)(i) · 2 MARKS

When does $(A+B)^2 = A^2 + 2AB + B^2$?

(a) Let A and B be two square matrices of the same size. Then:

(i) Show that $(A+B)^2 = A^2 + 2AB + B^2$ if $(A+B)(A-B) = A^2 - B^2$.

SOLUTION

Step 1. Expand the hypothesis, keeping the order of every product:

$$(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2 \;\Rightarrow\; BA - AB = O \;\Rightarrow\; AB = BA$$

Step 2. Expand the square and replace BA by AB:

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$$

In general $(A+B)^2 = A^2 + AB + BA + B^2$. Both identities in this question hold exactly when $AB = BA$.

SEMESTER 472 · QUESTION 2 (a)(ii) · 2 MARKS

A pair of matrices that do not commute

(a) Let A and B be two square matrices of the same size. Then:

(ii) Give an example of matrices A and B for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

SOLUTION

By part (i), any pair with $AB \neq BA$ works. Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, so $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and

$$(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

With $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$:

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} \neq \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

The difference is $BA - AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$: the cross terms that do not combine.

All right inverses of a 2×3 matrix

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find all matrices B such that $AB = I$.

SOLUTION

B must be 3×2, say $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$; its columns solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Reduce $[A \mid I_2]$ once for both:

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-2)}} \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -2 & 1 \end{array}\right] \xrightarrow{M_2^{(-1)},\, A_{2,1}^{(-1)}} \left[\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 2 & 2 & -1 \end{array}\right]$$

Column 1: $x_1 - x_3 = -1$, $x_2 + 2x_3 = 2$. With $x_3 = s$: $x_1 = -1 + s$, $x_2 = 2 - 2s$.

Column 2: $y_1 - y_3 = 1$, $y_2 + 2y_3 = -1$. With $y_3 = t$: $y_1 = 1 + t$, $y_2 = -1 - 2t$.

$B = \begin{bmatrix} -1 + s & 1 + t \\ 2 - 2s & -1 - 2t \\ s & t \end{bmatrix}$, $s, t \in \mathbb{R}$: infinitely many right inverses.

A patterned determinant: $(a+b)(a-b)^3$

(c) Show that $\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = (a+b)(a-b)^3$.

SOLUTION

Add row 3 to row 1 (no change in the determinant) and take out the common factor $a+b$. Then clear column 1 below the pivot; the result is triangular, so multiply its diagonal:

$$\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} \overset{A_{3,1}^{(1)}}{=} \begin{vmatrix} a+b & a+b & a+b & a+b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = (a+b) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix}$$

$$\overset{A_{1,2}^{(-b)}, A_{1,3}^{(-b)}, A_{1,4}^{(-b)}}{=} (a+b) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-b & a-b & 0 \\ 0 & 0 & a-b & a-b \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+b) \cdot 1 \cdot (a-b)^3$$

Two linear systems

(a) Find the value(s) of α such that the following linear system has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions: (6 marks)

$$\begin{aligned} x + 2y - z &= 2 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \\ x - 2y + 3z &= 1 \end{aligned}$$

(b) Consider the following system of linear equations: (4 marks)

$$\begin{aligned} x + 2y - 3z &= 1 \\ x + y - z &= 2 \\ x + y + 2z &= 3 \end{aligned}$$

(i) Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ by using $\text{adj}(A)$. (2.5 marks)

(ii) Use A^{-1} to solve the above linear system. (1.5 marks)

SEMESTER 472 · QUESTION 3 (a) · 6 MARKS

A parameter decides: none, one or infinitely many

(a) Find the value(s) of α such that the following linear system has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions:

$$\begin{aligned} x &+ 2y &- z &= 2 \\ x &+ 2y &- (\alpha^2-3)z &= \alpha \\ x &- 2y &+ 3z &= 1 \end{aligned}$$

SOLUTION

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 1 & 2 & 3-\alpha^2 & | & \alpha \\ 1 & -2 & 3 & | & 1 \end{bmatrix} \overset{A_{1,2}^{(-1)},\, A_{1,3}^{(-1)}}{\rightarrow} \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & 0 & 4-\alpha^2 & | & \alpha-2 \\ 0 & -4 & 4 & | & -1 \end{bmatrix} \overset{I_{2,3}}{\rightarrow} \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -4 & 4 & | & -1 \\ 0 & 0 & 4-\alpha^2 & | & \alpha-2 \end{bmatrix}$$

The last row says $(4-\alpha^2)z = \alpha - 2$, that is, $(2-\alpha)(2+\alpha)z = \alpha - 2$.

	$\alpha = -2$	$\alpha = 2$	$\alpha \neq \pm 2$
Last equation	$0z = -4$	$0z = 0$	$z = -1/(\alpha+2)$
Pivots	row 3 inconsistent	no pivot for z: free	a pivot in every column
Solutions	(i) none	(iii) infinitely many	(ii) exactly one

(i) $\alpha = -2$ (ii) $\alpha \in \mathbb{R}\backslash\{-2,2\}$ (iii) $\alpha = 2$

The inverse by the adjoint

(b) Consider the following system of linear equations:

$$\begin{aligned} x &+ 2y - 3z = 1 \\ x &+ y - z = 2 \\ x &+ y + 2z = 3 \end{aligned}$$

(i) Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ by using $\text{adj}(A)$.

SOLUTION

Along row 1: $|A| = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 3 - 6 + 0 = -3.$

Cofactors $C_{ij} = (-1)^{i+j} M_{ij}$, for example $C_{21} = -(4+3) = -7$ and $C_{32} = -(-1+3) = -2$:

$$C = \begin{bmatrix} 3 & -3 & 0 \\ -7 & 5 & 1 \\ 1 & -2 & -1 \end{bmatrix}, \qquad \text{adj}(A) = C^T = \begin{bmatrix} 3 & -7 & 1 \\ -3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = -\frac{1}{3}\begin{bmatrix} 3 & -7 & 1 \\ -3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 7/3 & -1/3 \\ 1 & -5/3 & 2/3 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

Solving the system with A^{-1}

(b) Consider the following system of linear equations:

(ii) Use A^{-1} to solve the above linear system.

$$\begin{aligned} x + 2y - 3z &= 1 \\ x + y - z &= 2 \\ x + y + 2z &= 3 \end{aligned}$$

SOLUTION

The system is $AX = B$ with $B = [\,1 \;\; 2 \;\; 3\,]^T$, and A is invertible, so $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 3 & -7 & 1 \\ -3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -8 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

Check in the first equation: $\frac{8}{3} + 2\left(-\frac{1}{3}\right) - 3\left(\frac{1}{3}\right) = \frac{8-2-3}{3} = 1$.

Solution: $x = \frac{8}{3}$, $y = -\frac{1}{3}$, $z = \frac{1}{3}$.