King Saud University: Mathematics Department Math-254
Second Semester 1445 H Second Midterm Exam. Solution
Maximum Marks = 25 Time: 90 mins.

Question 1: [6 Marks]

Use Gaussian elimination with partial pivoting to show that the following system is nonsingular and then compute the unique solution of the system using backward substitution.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0.5 \\ 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ -x_1 & - & 1.5x_2 & + & 2.5x_3 & = & -1 \end{array}$$

Solution. For the first elimination step, since 2 is the largest absolute coefficient of variable x_1 in the given system, therefore, the first row and the second row are interchange, we get

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ x_1 & + & x_2 & + & x_3 & = & 0.5 \\ -x_1 & - & 1.5x_2 & + & 2.5x_3 & = & -1 \end{array}$$

Then eliminate first variable x_1 from the second and the third rows by subtracting the multiples $m_{21} = 0.5$ and $m_{31} = -0.5$ of row 1 from rows 2 and 3 respectively, gives

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ & + & 2.5x_2 & + & 0.5x_3 & = & 1 \\ & - & 3x_2 & + & 3x_3 & = & -1.5 \end{array}$$

For the second elimination step, -3 is the largest absolute coefficient of second variable x_2 in the third row, so the second and third rows are interchange, giving us

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ & - & 3x_2 & + & 3x_3 & = & -1.5 \\ & + & 2.5x_2 & + & 0.5x_3 & = & 1 \end{array}$$

Eliminate first variable x_2 from the third row by subtracting the multiple $m_{32} = -0.8333$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ & - & 3x_2 & + & 3x_3 & = & -1.5 \\ & & & + & 3x_3 & = & -0.25 \end{array}$$

The original set of equations has been transformed to an equivalent upper-triangular form. Since we changed rows two times, so

$$\det(A) = (-1)^2[(2)(-3)(3)] = -18 \neq 0,$$

so the system is nonsingular.

Now use the transformed equivalent upper-triangular linear system form and using backward substitution, we get

$$x_1 = 0.1667 = 1/6, \qquad x_2 = 0.4167 = 5/12, \qquad x_3 = -0.0833 = -1/12,$$

the required unique solution of the given linear system. •

Question 2: [7 Marks]

Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Use LU decomposition by Dollittle's method to find the determinant of the matrix A and the unique solution of the linear system.

Solution. Since we know that

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = \frac{2\alpha}{\alpha} = 2 = l_{21}$, $m_{31} = \frac{1}{\alpha} = l_{31}$, and $m_{32} = \dfrac{(3\alpha - 4)}{(-9\alpha)} = l_{32}(\alpha \neq 0)$, gives

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & \frac{(3\alpha-4)}{\alpha} & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix},$$

which is the required LU decomposition of A by Dollittle's method.
The determinant of the matrix A can be obtained as

$$det(A) = det(U) = \frac{-9\alpha(\alpha^2 - 1)}{\alpha} = -9(\alpha^2 - 1).$$

Now to find the unique solution for system for nonzero α ($\neq \pm 1$) we do as follows:

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} = \mathbf{b}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, \frac{2\alpha-2}{\alpha}]^T = [6, -9, \frac{2(\alpha-1)}{\alpha}]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ \frac{2\alpha-2}{\alpha} \end{pmatrix},$$

and performing backward substitution yields, $[x_1, x_2, x_3]^T = [\frac{2\alpha-2}{\alpha^2-1}, 1, \frac{2\alpha-2}{\alpha^2-1}]^T = [\frac{2}{\alpha+1}, 1, \frac{2}{\alpha+1}]^T$.●

Question 3: [6 Marks]

Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Find the matrix form of the Gauss-Seidel iterative method and then compute the number of iterations needed to get an accuracy within 10^{-4}, using Gauss-Seidel iterative method and $\mathbf{x}^{(0)} = [0.5, 0.5, 0.5]^T$.

Solution. Let convert

$$A = L + U + D = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The Gauss-Seidel iteration matrix T_G is defined as

$$T_G = -(D + L)^{-1}U == \begin{pmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{pmatrix},$$

and the constant term

$$C_G = \begin{pmatrix} 1/5 \\ 11/15 \\ 71/60 \end{pmatrix}.$$

So the matrix form of Gauss-Seidel iterative method is

$$\mathbf{x}^{(k+1)} = \begin{pmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{pmatrix} \mathbf{x}^{(k)} + \begin{pmatrix} 1/5 \\ 11/15 \\ 71/60 \end{pmatrix}, \quad k \geq 0.$$

Using the matrix form of Gauss-Seidel iterative method for $k = 0$ and initial approximation, we get

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 & 0 & 1/5 \\ 0 & 0 & 1/15 \\ 0 & 0 & 1/60 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 1/5 \\ 11/15 \\ 71/60 \end{pmatrix} = \begin{pmatrix} 0.3000 \\ 0.7667 \\ 1.1917 \end{pmatrix},$$

and the l_∞-norm of the matrix T_G is

$$\|T_G\|_\infty = \max\left\{\frac{1}{5}, \frac{1}{15}, \frac{1}{60}\right\} = \frac{1}{5} = 0.2000,$$

we obtain

$$M = \frac{\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|}{1 - \|T_G\|} = \frac{0.6917}{(1 - 0.2000)} = 0.8646,$$

and

$$k \geq \frac{\ln(10^{-4}/(0.8646))}{\ln(0.2000)} = 5.6323, \quad \text{gives} \quad k = 6.$$

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Question 4: [6 Marks]

Consider the following linear system:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & -2 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 6 \\ 2x_1 & - & x_2 & - & x_3 & = & 1 \end{array}$$

If $\mathbf{x}^* = [1.01, 2.01, -0.98]^T$ is an approximate solution of the given linear system, then find the corresponding residual vector $\mathbf{r}$ and estimate the relative error $\dfrac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|}$.

Solution. The given system coefficients matrix is:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix},$$

and its inverse is given by:

$$A^{-1} = \begin{pmatrix} 0.6 & 0.4 & -0.2 \\ 0.4 & 0.6 & -0.8 \\ 0.8 & 0.2 & -0.6 \end{pmatrix}.$$

Thus, the l_∞-norms of these matrices are $\|A\|_\infty = 5$ and $\|A^{-1}\|_\infty = 1.8$, respectively. Hence, the condition number of A is:

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (5)(1.8) = 9.$$

The residual vector corresponding to $\mathbf{x}^* = [1.01, 2.01, -0.98]^T$ is given as:

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1.01 \\ 2.01 \\ -0.98 \end{pmatrix} = \begin{pmatrix} -0.02 \\ 0.01 \\ 0.01 \end{pmatrix}.$$

Therefore,

$$\|\mathbf{r}\|_\infty = 0.02.$$

From relative error formula, we have:

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq K(A) \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

Then, using the results in parts (a) and (b) and $\|\mathbf{b}\|_\infty = 6$, implies:

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq (9)\frac{(0.02)}{6} = 0.03.$$

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