King Saud University
College of Science
Department of Mathematics
$2^{nd}$ Sem. 1447 H. (25/26)

Math 380
Stochastic Processes
Second Midterm Exam
Time: 2 hours

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**Problem 1** [5 marks]:
Consider the Markov chain $X_0, X_1, \ldots$, with transition probability matrix given by

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0 & 0.4 & 0.6 \\ 0.7 & 0 & 0.3 \end{pmatrix} \end{array}.$$

Answer the following questions independently:

1. If the initial distribution is $\mathbb{p}_0 = \mathbb{P}\{X_0 = 0\} = 0.4$, $\mathbb{p}_1 = \mathbb{P}\{X_0 = 1\} = 0.3$, and $\mathbb{p}_2 = \mathbb{P}\{X_0 = 2\} = 0.3$, then calculate the following probabilities:
   (a) $\mathbb{P}\{X_0 = 0, X_1 = 1, X_2 = 2\}$.
   (b) $\mathbb{P}\{X_0 = 1, X_1 = 1, X_3 = 0\}$.
2. Determine the conditional probability $\mathbb{P}\{X_1 = 1, X_2 = 1 \big| X_0 = 0\}$.
3. If it is known that the process starts in state $X_0 = 2$, then determine the following probabilities:
   (a) $\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$.
   (b) $\mathbb{P}\{X_2 = 1\}$.

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**Problem 2** [5 marks]:
Consider the spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with independent demand random variables $\xi_n$ with probabilities $\mathbb{P}\{\xi_n = 0\} = 0.5$, $\mathbb{P}\{\xi_n = 1\} = 0.3$, $\mathbb{P}\{\xi_n = 2\} = 0.2$, and suppose that $s = 0$ and $S = 2$. Let a random variable $X_n$ be defined to be the quantity on hand at the end-of-period $n$.

1. Write the transition probability matrix of the Markov chain $\{X_n\}$.
2. What is the long run fraction of quantities in each one of this model's states.
3. Suppose that every period that this process spends in state $-1$ incurs a cost of \$1600. Every period that the process spends in state 0 incurs a cost of \$1600. Every period that the process spends in state 1 incurs a cost of \$0. Every period that the process spends in state 2 incurs a cost of \$0. What is the long run mean cost per period associated with this Markov chain?

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**Problem 3** [5 marks]:
An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. Assume that, within one year, the following happens:

- No customer is upgraded from poor to preferred category.



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- No customer is downgraded from preferred to poor category.
- 50% of poor category become satisfactory.
- 25% of satisfactory customers move to preferred category.
- 10% of satisfactory customers become poor.
- 15% of preferred customers are downgraded to satisfactory.

Then, answer the following questions:

1. Write the associated transition probability matrix.
2. What is the long run fraction of drivers in each of these three categories.

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**Problem 4** [5 marks]:
Consider the following transition probability matrix :

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0.2 & 0.3 & \beta & 0.1 \\ 0.3 & 0.1 & 0.5 & \gamma \\ 0 & \lambda & 0 & 1 \end{pmatrix} \end{array}.$$

1. Find the values of the constants $\alpha$, $\beta$, $\gamma$, $\lambda$, such that $\mathbf{P}$ defines a transition probability matrix of some Markov chain.
2. Assuming the values of $\alpha, \beta, \gamma, \lambda$ obtained in (1), sketch the transition diagram, and classify the states (as absorbing or communicating).
3. Starting in state $X_0 = 1$, determine the probability that the Markov chain ends in state 0.
4. If the process starts in state $X_0 = 1$ and gets absorbed in state 0, determine the mean time to absorption.

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**Problem 5** [5 marks]:
Consider the following stock market model, where the price of the stock is recorded at the end of a given day, assuming that the stock's value will be increased tomorrow depending upon whether it has increased today and yesterday. If the stock has increased for the last two days, then it will increase tomorrow with probability 0.7. If the stock has increased today but has decreased yesterday, then it will increase tomorrow with probability 0.6. If the stock has decreased today but has increased yesterday, then it will increase tomorrow with probability 0.4. If the stock has decreased for the last two days, then it will increase tomorrow with probability 0.3.

1. Transform this model into a Markov chain. Then, find its transition probability matrix, and sketch the corresponding transition diagram.
2. Find the long run fraction of days in which the value of the stock increases.
3. Find the long run fraction of days in which the value of the stock decreases.

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### Good Luck ###



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# Math 380

## second semester 2025/2026

## Model answers to Midterm exam 2

### problem 1 :

1) a) $P[X_0=0, X_1=1, X_2=2] = p_0 p_{01} p_{12} = (0.4)(0.3)(0.6) = \boxed{0.072}$

b) $P[X_0=1, X_1=1, X_3=0] = P[X_0=1, X_1=1, X_2=0, X_3=0] + P[X_0=1, X_1=1, X_2=1, X_3=0] + P[X_0=1, X_1=1, X_2=2, X_3=0]$.

$= p_1 p_{11} p_{10} p_{00} + p_1 p_{11} p_{11} p_{10} + p_1 p_{11} p_{12} p_{20}$

$= (0.3)(0.4)(0)(0.2) + (0.3)(0.4)(0.4)(0) + (0.3)(0.4)(0.6)(0.7)$

$= 0 + 0 + 0.0504 = \boxed{0.0504}$

2) $P[X_1=1, X_2=1 \mid X_0=0] = \dfrac{P[X_0=0, X_1=1, X_2=1)}{P(X_0=0)} = \dfrac{p_0 p_{01} p_{11}}{p_0} = (0.3)(0.4)$

$= \boxed{0.12}$.

3) $X_0=2$ is known, so $P(X_0=2)=1$, $P(X_0=0)=P(X_0=1)=0$.

$P(X_0=2) = p_2$, $P(X_0=0) = p_0$, $P(X_0=1) = p_1$

a) $P[X_0=2, X_1=1, X_2=0] = p_2 p_{21} p_{10} = (1)(0)(0) = \boxed{0}$.

b) $P[X_2=1] = P[X_0=0, X_2=1] + P[X_0=1, X_2=1] + P[X_0=2, X_2=1]$ (2 steps, 2 steps, 2-steps)

$= p_0 p_{01}^{(2)} + p_1 p_{11}^{(2)} + p_2 p_{21}^{(2)}$

$= (0)(p_{01}^{(2)}) + (0) p_{11}^{(2)} + (1) p_{21}^{(2)} = p_{21}^{(2)}$.

$= (0.7)(0.3) + (0)(0.4) + (0.3)(0)$.

$= \boxed{0.21}$

### problem 2 :

1) The possible values of $X_n$ are $S=2, 1, 0, -1$ as $S=2, s=0$

The model entries are

$p_{ij} = P[X_{n+1}=j \mid X_n=i] = \begin{cases} P(\xi_{n+1}=i-j), & 0<i\leq 2 \text{ (no replen} \\ P(\xi_{n+1}=2-j), & i\leq 0, \text{ (replenish} \end{cases}$

So we get the matrix

$$P = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0.2 & 0.3 & 0.5 \\ 0 & 0.2 & 0.3 & 0.5 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \end{pmatrix} \end{array}$$

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2) $\pi P = \pi \iff (\pi_{-1}, \pi_0, \pi_1, \pi_2)\begin{pmatrix} 0 & 0.2 & 0.3 & 0.5 \\ 0 & 0.2 & 0.3 & 0.5 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \end{pmatrix} = (\pi_{-1}, \pi_0, \pi_1, \pi_2)$

↳ or $P^T \pi^T = \pi^T$

We get the equations

$$\begin{cases} 0.2\,\pi_1 = \pi_{-1} \\ 0.2\,\pi_{-1} + 0.2\,\pi_0 + 0.3\,\pi_1 + 0.2\,\pi_2 = \pi_0 \\ \cancel{0.3\,\pi_{-1} + 0.3\,\pi_0 + 0.5\,\pi_1 + 0.3\,\pi_2 = \pi_1} \text{ strike this one} \\ 0.5\,\pi_{-1} + 0.5\,\pi_0 + 0.5\,\pi_2 = \pi_2 \end{cases}$$

together with $\pi_{-1} + \pi_0 + \pi_1 + \pi_2 = 1$

multiplying by 10, we get

$$\begin{cases} \pi_1 = 5\,\pi_{-1} & (1) \\ 2\,\pi_{-1} - 8\,\pi_0 + 3\,\pi_1 + 2\,\pi_2 = 0 & (2) \\ 5\,\pi_{-1} + 5\,\pi_0 - 5\,\pi_2 = 0 & (3) \\ \pi_{-1} + \pi_0 + \pi_1 + \pi_2 = 1 & (4) \end{cases}$$

$5 \times (2) + 2 \times (3) \Rightarrow 20\,\pi_{-1} - 30\,\pi_0 + 15\,\pi_1 = 0$

$\pi_1 = 5\pi_{-1}$ from (1)

$$\Rightarrow \pi_0 = \frac{20\pi_{-1} + 75\pi_{-1}}{30} = \frac{95}{30}\pi_{-1} \Rightarrow \pi_0 = \frac{19}{6}\pi_{-1}$$

$$(3) \Rightarrow \pi_2 = \pi_0 + \pi_{-1} = \left(\frac{19}{6} + 1\right)\pi_{-1} \Rightarrow \pi_2 = \frac{25}{6}\pi_{-1}$$

Substituting in (4), we obtain

$$\pi_{-1} + \frac{19}{6}\pi_{-1} + 5\pi_{-1} + \frac{25}{6}\pi_{-1} = 1$$

$$\Rightarrow \pi_{-1}\left(1 + \frac{19}{6} + 5 + \frac{25}{6}\right) = 1 \Rightarrow \pi_{-1} = \frac{3}{40}$$

$$\Rightarrow \pi_0 = \frac{19}{6}\pi_{-1} = \frac{19}{80},\ \pi_1 = 5\pi_{-1} = \frac{3}{8},\ \pi_2 = \frac{25}{6}\pi_{-1} = \frac{5}{16}$$

$$\Rightarrow \boxed{(\pi_{-1}, \pi_0, \pi_1, \pi_2) = \left(\frac{3}{40}, \frac{19}{80}, \frac{3}{8}, \frac{5}{16}\right).}$$

3) $C = \sum_{j=0}^{N} \pi_j C_j \Rightarrow C = \pi_{-1}C_{-1} + \pi_0 C_0 + \pi_1 C_1 + \pi_2 C_2$

$$C = \left(\frac{3}{40}\right)(1600) + \left(\frac{19}{80}\right)(1600) + \left(\frac{3}{8}\right)(0) + \left(\frac{5}{16}\right)(0)$$

$$= 3 \times 40 + 19 \times 20 = 500$$

$$\Rightarrow \boxed{C = 500\ \$}$$

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## problem 3:

1) Let us label the states $\begin{cases} 0 \text{ poor customers} \\ 1 \text{ satisfactory customers} \\ 2 \text{ preferred customers} \end{cases}$

Thus the transition probability matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.1 & 0.65 & 0.25 \\ 0 & 0.15 & 0.85 \end{pmatrix} \end{array}$$

2) $\pi P = \pi \Leftrightarrow P^T \pi^T = \pi^T$, so we get the equations

$$\begin{cases} 0.5\,\pi_0 + 0.1\,\pi_1 = \pi_0 \\ \cancel{0.5\,\pi_0 + 0.65\,\pi_1 + 0.15\,\pi_2 = \pi_1} \\ 0.25\,\pi_1 + 0.85\,\pi_2 = \pi_2 \end{cases}$$ strike this equation

with

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Multiplying by 10 for the 1st equation and by 100 for the second, we get

$$\begin{cases} \pi_1 = 5\,\pi_0 \\ 15\,\pi_2 = 25\,\pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_0 = \frac{1}{5}\,\pi_1 \\ \pi_2 = \frac{5}{3}\,\pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \left(\frac{1}{5} + 1 + \frac{5}{3}\right)\pi_1 = 1 \Rightarrow \pi_1 = \frac{15}{43} \Rightarrow \pi_0 = \frac{3}{43},\ \pi_2 = \frac{25}{43}$$

$$\Rightarrow (\pi_0, \pi_1, \pi_2) = \left(\frac{3}{43}, \frac{15}{43}, \frac{25}{43}\right)$$

poor drivers $\longrightarrow \pi_0 = \frac{3}{43}$

satisfactory drivers $\longrightarrow \pi_1 = \frac{15}{43}$

preferred drivers $\longrightarrow \pi_2 = \frac{25}{43}$

## problem 4:

1) $\alpha + 0 + 0 + 0 = 1 \Rightarrow \alpha = 1$

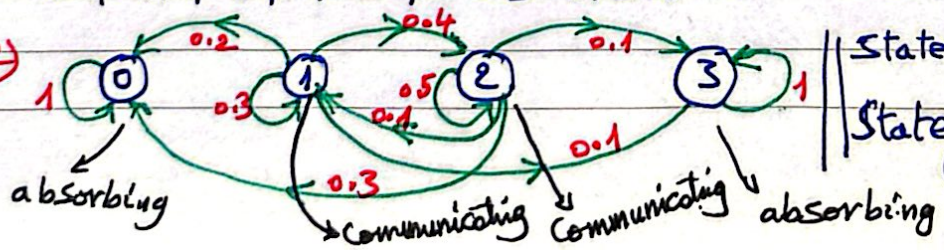
$0.2 + 0.3 + \beta + 0.1 = 1 \Rightarrow \beta = 0.4$

$0.3 + 0.1 + 0.5 + \gamma = 1 \Rightarrow \gamma = 0.1$

$0 + \lambda + 0 + 1 = 1 \Rightarrow \lambda = 0$

$$\Rightarrow P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0.1 & 0.5 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

2)




States 0 and 3 absorbing

States 1 and 2 communicating (transient)

3) put $u_1 = P\{X_T = 0 \mid X_0 = 1\}$ and $u_2 = P\{X_T = 0 \mid X_0 = 2\}$

The first step analysis yields the equations

$$\begin{cases} u_1 = 0.2 + 0.3u_1 + 0.4u_2 \\ u_2 = 0.3 + 0.1u_1 + 0.5u_2 \end{cases} \Leftrightarrow \begin{cases} 7u_1 - 4u_2 = 2 \\ -u_1 + 5u_2 = 3 \end{cases}$$

Solving, we get $\boxed{u_2 = \frac{23}{31}}$ and $\boxed{u_1 = \frac{22}{31}}$

so the answer is $\boxed{u_1 = \frac{22}{31}}$

4) put $v_1 = E[T \mid X_0 = 1]$ and $v_2 = E[T \mid X_0 = 2]$.

Then, we get the equations:

$$\begin{cases} v_1 = 1 + 0.3v_1 + 0.4v_2 \\ v_2 = 1 + 0.1v_1 + 0.5v_2 \end{cases} \Leftrightarrow \begin{cases} 7v_1 - 4v_2 = 10 \\ -v_1 + 5v_2 = 10 \end{cases}$$

$$\Rightarrow \boxed{v_1 = \frac{90}{31}}, \boxed{v_2 = \frac{80}{31}}$$

so the answer is $\boxed{v_1 = \frac{90}{31}}$

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problem 5+

1) We label the states

$(i,i)$ → stock increased both today and yesterday.

$(i,d)$ → stock increased yesterday and decreased today.

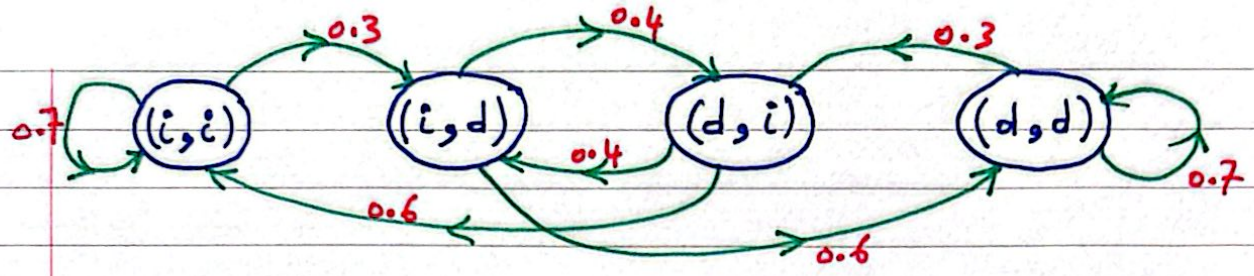
$(d,i)$ → stock decreased yesterday and increased today

$(d,d)$ → stock decreased both yesterday and today.

so the transition probability matrix is:

$$P = \begin{array}{c} \\ (i,i) \\ (i,d) \\ (d,i) \\ (d,d) \end{array} \begin{array}{c} \begin{array}{cccc} (i,i) & (i,d) & (d,i) & (d,d) \end{array} \\ \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix} \end{array}$$

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2) The equations determining $(\pi_0, \pi_1, \pi_2, \pi_3)$ are

$$\begin{cases} 0.7\pi_0 + 0.6\pi_2 = \pi_0 \\ 0.3\pi_0 + 0.4\pi_2 = \pi_1 \\ 0.4\pi_1 + 0.3\pi_3 = \pi_2 \\ 0.6\pi_1 + 0.7\pi_3 = \pi_3 \quad \text{strike this equation} \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Multiplying by 10, we get the equations

$$\boxed{\pi_2 = \tfrac{1}{2}\pi_0} \quad (1)$$

$$10\pi_1 = 4\pi_2 + 3\pi_0 = \tfrac{4}{2}\pi_0 + 3\pi_0 = 5\pi_0 \Rightarrow \boxed{\pi_1 = \tfrac{1}{2}\pi_0} \quad (2)$$

$$10\pi_2 = 4\pi_1 + 3\pi_3 \Rightarrow \pi_3 = \tfrac{1}{3}(10\pi_2 - 4\pi_1) \quad (3)$$

(1) and (2) and (3) $\Rightarrow$ $\pi_3 = \tfrac{5}{3}\pi_0 - \tfrac{2}{3}\pi_0$

$$\Rightarrow \boxed{\pi_3 = \pi_0}$$

Substituting in (*), we get

$$\pi_0 + \tfrac{1}{2}\pi_0 + \tfrac{1}{2}\pi_0 + \pi_0 = 1 \Rightarrow \boxed{\pi_0 = \tfrac{1}{3}}$$

$$\boxed{\pi_3 = \pi_0 = \tfrac{1}{3}}, \quad \boxed{\pi_1 = \pi_2 = \tfrac{1}{2}\pi_0 = \tfrac{1}{6}}$$

$$\Rightarrow \boxed{(\pi_0, \pi_1, \pi_2, \pi_3) = \left(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}\right)}$$

So the fraction of days in which the stock increases is

$$\boxed{\pi_0 + \pi_1 = \tfrac{1}{3} + \tfrac{1}{6} = \tfrac{1}{2}.}$$

3) By the latter calculation, the fraction of days in which the stock decreases is

$$\boxed{\pi_2 + \pi_3 = \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{2}}$$

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