King Saud University

College of Sciences

Department of Mathematics

Solution of Mid-term 2	Math 481	Semester I - 1445

Question 1 :

1. $\left|\frac{1}{n}\sin(\frac{x}{n})\right| \leq \frac{|x|}{n^2}$, then the series $\sum_{n\geq 1}\frac{1}{n}\sin(\frac{x}{n})$ converges absolutely on $\mathbb{R}$ and normally on any compact of $\mathbb{R}$.
2. If $f_n(x) = \frac{x^n}{1+x^{2n}}$, then $f_n(x) = f_n(\frac{1}{x})$, for $x \neq 0$.
   $\lim_{n\to+\infty} f_n(x) = 0$ for $|x| < 1$ and $\lim_{n\to+\infty} f_n(1) = \frac{1}{2}$.
   For $|x| < 1$, the series $\sum_{n\geq 1}\frac{x^n}{1+x^{2n}}$ converges absolutely and normally on any compact of $(-1, 1)$.

Question 2 :

By parts: $\int_0^1 \tan^{-1} x dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$. $\tan^{-1} x = \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$.
$\left|\sum_{k=n}^{m}\frac{(-1)^k x^{2k+1}}{2k+1}\right| \leq \frac{1}{2n+1}$. Then $\int_0^1 \tan^{-1} x dx = \sum_{n=0}^{+\infty}\int_0^1 \frac{(-1)^n x^{2n+1}}{2n+1}dx = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)(2n+2)}$.

Question 3 :

1. The series $\sum_{n=1}^{+\infty}\frac{(-1)^n e^{nx}}{n^2}$ converges only for $x \leq 0$. $g(x) = \sum_{n=1}^{+\infty}\frac{(-1)^n e^{nx}}{n^2}$
2. The series $\sum_{n=1}^{+\infty}\frac{(-1)^n e^{nx}}{n^2}$ converges on D, the series $\sum_{n=1}^{+\infty}\frac{(-1)^n e^{nx}}{n}$ converges uniformly on D because $\left|\sum_{k=n}^{m}\frac{(-1)^k e^{kx}}{k}\right| \leq \frac{1}{n}$, the g is $\mathscr{C}^1$) on D.

Question 4 :

1. The series $\sum_{n\geq 1}(-1)^n \frac{x^{n+1}}{n^2}$ is a power series, then it is $\mathscr{C}^\infty$ on $[-1,1]$.

2. For $x \neq 0$, let $h(x) = \sum_{n=1}^{+\infty}(-1)^n \frac{x^n}{n^2}$, $h'(x) = \sum_{n=1}^{+\infty}(-1)^n \frac{x^{n-1}}{n} = -\frac{1}{x}\ln(1+x)$. $f(x) = x\int_0^x \frac{1}{t}\ln(1+t)dt$.