MID TERM EXAMINATION, SEMESTER II, 2024
Dept. Math., College of Science, KSU
Math: 107 Full Mark: 25 Time: 90 Min.

Q1.[5] Find the values of δ for which the following linear system of equations

$$\begin{aligned} x + y + z + t &= 4 \\ x + \delta y + z + t &= 4 \\ x + y + \delta z + (3 - \delta)t &= 6 \\ 2x + 2y + 2z + (\delta - 5)t &= 6 \end{aligned}$$

has (i) no solution (ii) infinitely many solutions.

Q2.(a) [4+3=7] If

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

Find $A + A^T + A^{-1}$.

(b) If the inverse of $2A$ is

$$\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

Find the matrix A.

Q3 [5] For the following linear system of equations:

$$\begin{aligned} x - z &= 6 \\ x + y + z &= -3 \\ -x + y &= 12 \end{aligned}$$

Find the inverse of the coefficient matrix by using elementary row operations, then find the solution of the given system.

Q4. [4] Evaluate the determinant of the matrix by reducing the matrix to row echelon form

$$A = \begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Q5. [4] Use Cramer's rule to solve the following linear system of equations:

$$\begin{aligned} x + y &= 1 \\ x + 2y + z &= -1 \\ x + 3y - z &= 2. \end{aligned}$$

Question 1: Find the values of δ for which the following linear system of equations

$$
\begin{aligned}
x + y + z + t &= 4 \\
x + \delta y + z + t &= 4 \\
x + y + \delta z + (3-\delta)\, t &= 6 \\
2x + 2y + 2z + (\delta - 5)\, t &= 6
\end{aligned}
$$

has: (*i*) no solution (*ii*) infinitely many solutions.

Solution: Augmented matrix of the given system $\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 1 & \delta & 1 & 1 & 4 \\ 1 & 1 & \delta & 3-\delta & 6 \\ 2 & 2 & 2 & \delta-5 & 6 \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & \delta-1 & 0 & 0 & 0 \\ 0 & 0 & \delta-1 & -5 & 0 \\ 0 & 0 & 0 & \delta-7 & -2 \end{array}\right]$.

Hence, the system has no solution if $\delta \in \{1,7\}$; and infinitely many solutions if $\delta \in \{\,\}$.

Q. 1 2,

(a) If $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, Find $\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \mathbf{A}^{-1}$

Answer

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \mathbf{A}^{-1} = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} + \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 \\ 7 & 13 \end{bmatrix}$$

(b) If the inverse of $2\mathbf{A}$ is $\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$, Find the matrix $\mathbf{A}$.

Answer

$$(2\mathbf{A})^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\frac{1}{2}(\mathbf{A})^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 4 & -4 \\ -6 & 10 \end{bmatrix}$$

$$\therefore \mathbf{A} = \frac{1}{16}\begin{bmatrix} 10 & 4 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5/8 & 1/4 \\ 3/8 & 1/4 \end{bmatrix}$$

Q. 2 3,

For the following linear system of equations: $x - z = 6,\ x + y + z = -3,\ -x + y = 12$

Find the inverse of the coefficient matrix by using elementary row operations,

then find the solution of the given system.

Answer

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\xrightarrow[R_1+R_3]{-R_1+R_2,} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -3 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1/3 & -1/3 \end{array}\right)$$

$$\xrightarrow[R_3+R_2]{R_3+R_1,}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & -2/3 & 1/3 & -1/3 \end{array}\right)$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix}$$

$\therefore$ **X=A^{-1}B**

$$= \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \\ -9 \end{pmatrix}$$

The solution is $x = -3, y = 9, z = -9$.

Q 4

$$\det(A) = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= (-1)(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1)(2)(1) = -2.$$

Question 5: Use Cramer's rule to solve the following linear system of equations:

$$\begin{aligned} x + y \quad &= 1 \\ x + 2y + z &= -1 \\ x + 3y - z &= 2. \end{aligned}$$

Solution: Let A denote the matrix of coefficients. Then, $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -3$. So, the Cramer's is applicable on the given system. Therefore, $= \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix}}{|A|} = \dfrac{-4}{-3} = \dfrac{4}{3}$; $y = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}}{|A|} = \dfrac{1}{-3} = \dfrac{-1}{3}$ and $z = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix}}{|A|} = \dfrac{5}{-3} = \dfrac{-5}{3}$.