First Midterm — Model Answer

MATH 111 — Integral Calculus

Section 481 • 11 October 2026 • Total 15 marks • Dr. Sumayah Mansour Bin Hazzaa

Marks for each step are shown on the right. Equivalent correct forms receive full marks. Every indefinite integral must carry the constant of integration.

Question I (2 marks) *Mean Value Theorem for definite integrals*

$f(x) = x^2 + 2$ on $[0, 3]$. The theorem guarantees a z in $[0, 3]$ at which $f(z)$ equals the average value of f over the interval:

$$f(z) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3}\int_0^3 (x^2+2)\,dx \qquad [0.5]$$

$$\int_0^3 (x^2+2)\,dx = \left[\frac{x^3}{3} + 2x\right]_0^3 = 9 + 6 = 15 \quad \Longrightarrow \quad f(z) = \frac{15}{3} = 5 \qquad [1]$$

Solve $f(z) = 5$:

$$z^2 + 2 = 5 \quad \Longrightarrow \quad z^2 = 3 \quad \Longrightarrow \quad z = \pm\sqrt{3}$$

$$z = \sqrt{3} \approx 1.73 \in [0, 3] \qquad (\ z = -\sqrt{3} \text{ is rejected: it is outside } [0, 3]\ ) \qquad [0.5]$$

Question II (2 marks) *logarithmic differentiation*

$y = \dfrac{e^{3x}\,\operatorname{csch}(x^4)}{\sqrt{\sin(3x)}}$. Take the natural logarithm of both sides first, then expand:

$$\ln y = 3x + \ln \operatorname{csch}(x^4) - \tfrac{1}{2}\ln(\sin(3x)) \qquad [0.5]$$

Differentiate implicitly. For the middle term use $\dfrac{d}{du}\ln\operatorname{csch} u = \dfrac{-\operatorname{csch} u \coth u}{\operatorname{csch} u} = -\coth u$, with $u = x^4$:

$$\frac{1}{y}\frac{dy}{dx} = 3 - 4x^3\coth(x^4) - \frac{3\cos(3x)}{2\sin(3x)} = 3 - 4x^3\coth(x^4) - \tfrac{3}{2}\cot(3x) \qquad [1]$$

Multiply by y:

$$\frac{dy}{dx} = \frac{e^{3x}\,\operatorname{csch}(x^4)}{\sqrt{\sin(3x)}}\left[\,3 - 4x^3\coth(x^4) - \tfrac{3}{2}\cot(3x)\,\right] \qquad [0.5]$$

Question III (11 marks) *evaluate the following integrals*

(1) $\displaystyle\int e^{\,1+\ln x}\,dx$ [1 mark]

Split the exponent before integrating: $e^{1+\ln x} = e^1 \cdot e^{\ln x} = e\,x$.

$$\int e^{\,1+\ln x}\,dx = e\int x\,dx = \boxed{\frac{e\,x^2}{2} + C}$$

(2) $\displaystyle\int \frac{3^x}{9+3^x}\,dx$ [1.5 marks]

Let $u = 9+3^x$, so $du = 3^x \ln 3\;dx$.

$$= \frac{1}{\ln 3}\int \frac{du}{u} = \frac{\ln|u|}{\ln 3} + C = \boxed{\frac{\ln(9+3^x)}{\ln 3} + C}$$

The absolute value is not needed: $9+3^x > 0$ for every x.

(3) $\displaystyle\int x^5\sqrt{x^3+2}\,dx$ [2 marks]

Let $u = x^3+2$, so $x^3 = u-2$ and $du = 3x^2dx$. Split $x^5dx = x^3 \cdot x^2dx$:

$$= \int (u-2)\sqrt{u}\,\frac{du}{3} = \frac{1}{3}\int \left(u^{3/2} - 2u^{1/2}\right)du = \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right] + C$$

$$= \frac{2}{15}\left(x^3+2\right)^{5/2} - \frac{4}{9}\left(x^3+2\right)^{3/2} + C$$

The substitution alone is not enough here: x^5 must be written as $x^3 \cdot x^2$, and the x^3 replaced by $u-2$.

(4) $f(x) = \begin{cases} x^2, & x < 3 \\ -5x, & x > 3 \end{cases}$, find $\displaystyle\int_0^4 f(x)\,dx$ [1.5 marks]

The rule changes at $x = 3$, so split the interval there:

$$\int_0^4 f(x)\,dx = \int_0^3 x^2\,dx + \int_3^4 (-5x)\,dx$$ [0.5]

$$= \left[\frac{x^3}{3}\right]_0^3 + \left[-\frac{5x^2}{2}\right]_3^4 = 9 + \left(-40 + \frac{45}{2}\right) = 9 - \frac{35}{2}$$

$$\int_0^4 f(x)\,dx = -\frac{17}{2} = -8.5$$ [1]

(5) $\displaystyle\int \frac{dx}{x^2-2x+5}$ [2 marks]

The denominator does not factor, so complete the square:

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x-1)^2 + 2^2$$ [1]

This is the inverse-tangent form $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$ with $u = x-1$ and $a = 2$:

$$= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C}$$ [1]

(6) $\displaystyle\int \frac{4^{\tanh x}}{\cosh^2 x}\,dx$ [1.5 marks]

Write $\dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$, which is exactly the derivative of $\tanh x$. Let $u = \tanh x$, so $du = \operatorname{sech}^2 x\,dx$, and the integral becomes an exponential in u:

$$\int \frac{4^{\tanh x}}{\cosh^2 x}\,dx = \int 4^{\tanh x}\operatorname{sech}^2 x\,dx = \int 4^u\,du \qquad [1]$$

Now use $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$ with $a = 4$:

$$= \frac{4^u}{\ln 4} + C = \boxed{\frac{4^{\tanh x}}{\ln 4} + C} \qquad [0.5]$$

The exponent is $\tanh x$, so this is an exponential integral, not a power. Reading it as $4\tanh x$ gives $2\tanh^2 x + C$, which is a different function.

(7) $\displaystyle\int \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$ [1.5 marks]

Let $u = \ln x$, so $du = \dfrac{dx}{x}$:

$$= \int \frac{u}{\sqrt{1-u^2}}\,du$$

Now a second substitution $w = 1 - u^2$, $dw = -2u\,du$:

$$= -\frac{1}{2}\int \frac{dw}{\sqrt{w}} = -\sqrt{w} + C = -\sqrt{1-u^2} + C$$

$$= \boxed{-\sqrt{1-(\ln x)^2} + C}$$

Note the $\ln x$ in the numerator: without it the integral would be $\sin^{-1}(\ln x) + C$. Students who reach for the inverse sine here lose the marks.