King Saud University, College of Sciences Mathematical Department.

Mid-Term Exam 2 (M425)/S1/2024
Full Mark:25. Time 1H30m
28/11/2024

Question 1. **[5,4]** **a)** Solve the Cauchy problem

$$xu_{xx} + 2x^2u_{xy} - u_x = 1,\ 1 < x < 0,\ y > 0.$$
$$u_x(1, y) = 0,\ u(1, y) = 1,\ y > 0.$$

b) What are the conditions on the constants a, b, c, d, e for which the function u is harmonic

$$i)\ u(x, y) = e^{ax} \cos(by) + e^{cx} \sin(dy)$$
$$ii)\ u(x, y, z) = x^2 + ay^2 + bz^2 + cxy + dxz + eyz$$

Question 2. **[4,4]** **a)** Assume that the function ψ is harmonic and non constant on $\mathbb{R}^2$. When the function $F \circ \psi$ can be harmonic?

b) Find the solution of the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0,\ 0 < x < \pi,\ 0 < y < 2\pi \\ u(0, y) = u(\pi, y) = 0,\ 0 < y < 2\pi \\ u(x, 0) = 0,\ u(x, 2\pi) = 1,\ 0 < x < \pi. \end{cases}$$

Question 3. **[4,4]** **a)** By using polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, show that the differential equation

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0,$$

can be written as $u_{rr} = 0$. Deduce the general solution of $u_{rr} = 0$, then write this solution in chartezian coordinates.

b) Find the maximum and minimum of the functions $u(x, y) = e^x \cos y$, and $v(x, y) = e^x \sin y$ on the square $[0, \pi] \times [0, \pi]$.