Q1 (a) a stochastic process Z_n is a martingale w.r.t $\mathcal{F}_n$ if $\forall n$

1. $E|Z_n| < \infty$
2. $E[Z_{n+1} | \mathcal{F}_n] = Z_n$

(b) $Z_n = \dfrac{Y_1 \cdots Y_n}{a_1 \cdots a_n}$, $E|Y_i| < \infty \ \forall i$ & $a_i = E[Y_i] \ \forall i$, constants

(1)

$$|Z_n| = \left| \frac{Y_1 \cdots Y_n}{a_1 \cdots a_n} \right|$$

$$= \frac{1}{|a_1 \cdots a_n|} |Y_1 \cdots Y_n|$$

$$= \frac{1}{|a_1 \cdots a_n|} |Y_1| \cdots |Y_n|$$

$$\Rightarrow E|Z_n| = \frac{1}{|a_1 \cdots a_n|} E\left[|Y_1| \cdots |Y_n|\right]$$

$$= \frac{1}{|a_1 \cdots a_n|} \underbrace{E|Y_1| \cdot E|Y_2| \cdots E|Y_n|}_{\text{as } Y_i\text{'s are independent}}$$

$$< \infty$$

as $a_i = E[Y_i]$ constants $\neq 0$ and $E|Y_i| < \infty \ \forall i$

3 2 $E[Z_{n+1} \mid F_n]$

$$= E\left[\frac{Y_1 \cdots Y_{n+1}}{a_1 \cdots a_{n+1}} \,\middle|\, F_n\right]$$

$$= \underbrace{\frac{Y_1 \cdots Y_n}{a_1 \cdots a_n}}_{F_n\text{-adapted}} E\left[\frac{Y_{n+1}}{a_{n+1}} \,\middle|\, F_n\right]$$

independent of F_n

$$= \frac{Y_1 \cdots Y_n}{a_1 \cdots a_n} E\left[\frac{Y_{n+1}}{a_{n+1}}\right]$$

$$= \frac{Y_1 \cdots Y_n}{a_1 \cdots a_n} \cdot \frac{1}{a_{n+1}} \cancel{E[Y_{n+1}]}$$

$$a_{n+1} = E[Y_{n+1}]$$

$$= \frac{Y_1 \cdots Y_n}{a_1 \cdots a_n} = Z_n$$

1 & 2 $\Longrightarrow$ Z_n is a martingale.

Q2: 2

(a) The fortune for player A is $i = \$5$ and the total amount is $N = \$5 + \$10 = \$15$

p=0.5071 $\Rightarrow$ q=0.4929

$u_i = pr\{X_n \text{ reaches state 0 before state } N | X_0 = i\}$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}, \ p \neq q$$

$$\therefore u_i = \frac{(0.4929/0.5071)^5 - (0.4929/0.5071)^{15}}{1-(0.4929/0.5071)^{15}}$$

$u_i = 0.61837$

(b) if $p = 0.5 \Rightarrow q = 1 - p = 0.5$

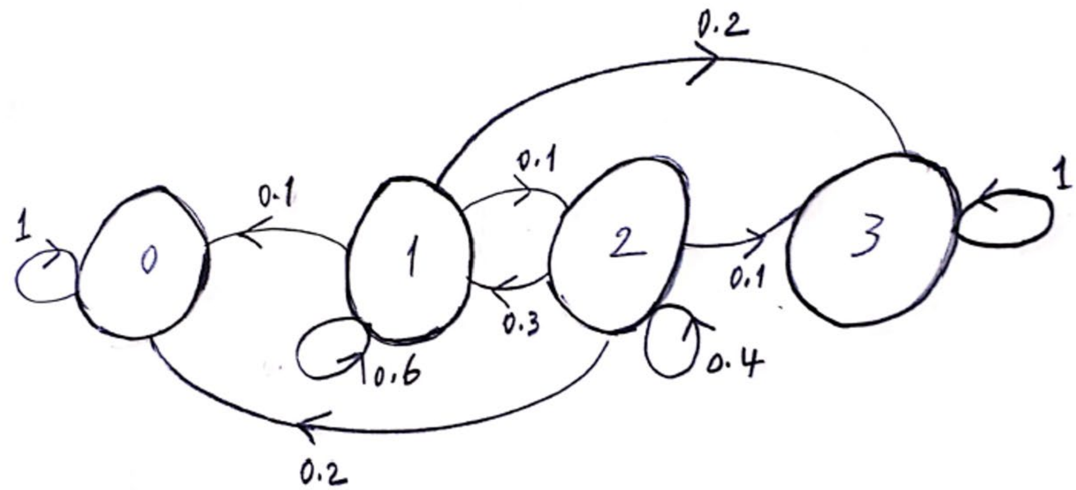
2

So that $u_i = \frac{N-i}{N} = \frac{15-5}{15} = \frac{10}{15}$

$= 0.667$

Q3:(a) (i) 1

It's an absorbing Markov Chain.



Markov Chain Diagram

(ii) 2

$$
\mathbf{P}=\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}
$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

$and\ \ v_i = \mathrm{E}[T | X_0 = i]$ for i=1,2.

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{11}{21}$$
$$\approx 0.52$$

(iii) Also, the mean time to absorption can be found as follows

1.5

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$\therefore$ The mean time to absorption is

$$v_1 = v_2 = \tfrac{10}{3}$$

$$\therefore\ v_2 = v_{20} = \tfrac{10}{3}$$
$$\approx 3.3$$

Q3:(b) 1.5

$$P_r\{X_2 = 1 \mid X_0 = 1\} = P_{11}^2 = 0.7756$$

$$P^2 = P \cdot P = \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix}$$

$$= \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{bmatrix} \end{array}$$

Q4:

(a) 2

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b) 3

i) $\text{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_0 = 1\}$

$= 0.5(0.2)(0.4)$

$= 0.04$

ii) $\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_1 = 1\}$

$\text{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$

$= P_{01} p_0 + P_{11} p_1 + P_{21} p_2$

$= 0.3(0.3) + 0.2(0.5) + 0.3(0.2) = 0.25$

$\therefore \text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.25(0.2)(0.4) = 0.02$

Q5: 6

$$\pi_j = \sum_{k=0}^{2} \pi_k P_{kj}$$

at $j=0$

$$\Rightarrow \pi_0 = 0.5\pi_0 + 0.5\pi_1 + 0.3\pi_2$$
$$\therefore 5\pi_0 - 5\pi_1 - 3\pi_2 = 0 \qquad (1)$$

at $j=1$

$$\Rightarrow \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2$$
$$\therefore 2\pi_0 - 9\pi_1 + 2\pi_2 = 0 \qquad (2)$$

and $\because \ \pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$

$\therefore$ By solving equations (1), (2) and (3)

We get $\pi_0 = 0.4205,\ \pi_1 = 0.1818,\ \pi_2 = 0.3977$

The long run mean cost per unit period is

$$C = \sum_{j=0}^{2} \pi_j c_j$$
$$= \pi_0 c_0 + \pi_1 c_1 + \pi_2 c_2$$
$$= \$\ 4.9431$$

Q6(a): (2

$$pr\{X_2 = 0\} = pr\{X_2 = 0|X_0 = 0\}\, pr\{X_0 = 0\} + pr\{X_2 = 0|X_0 = 1\}\, pr\{X_0 = 1\}$$

$$= p_{00}^{2}\, p_0 + p_{10}^{2}\, p_1$$

$$= (0.36)(0.4) + (0.32)(0.6) = 0.336\,,$$

vhere $p^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.24 & 0.4 \\ 0.32 & 0.28 & 0.4 \\ 0.32 & 0.24 & 0.44 \end{bmatrix}$

Q(b): 3

Given in Pb

Suppose that $s=0$ and $S=3$ and that the probability distribution for the demand is $\Pr\{\xi_n=0\}=0.1$, $\Pr\{\xi_n=1\}=0.4$, $\Pr\{\xi_n=2\}=0.3$ and $\Pr\{\xi_n=3\}=0.2$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n.

Ans:

The transition probability matrix is given by

$$\mathbf{P}=\begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccccc} 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \end{array}\right\| \end{array}$$

where,

$$P_{ij}=\Pr\{X_{n+1}=j \mid X_n=i\}$$
$$=\begin{cases} \Pr(\xi_{n+1}=3-j), \ i\le 0 & \text{replenishment} \\ \Pr(\xi_{n+1}=i-j), \ 0<i\le 3 & \text{without replenishment} \end{cases}$$