

Mid Term Exam Math 316

Full Marks 25, Duration 1H 30 (2025/2026)

Question 1.

a) Show that the set of functions: $f(x) = x$, $g(x) = |x|$, $h(x) = 1$ are linearly independent on $C[-1, 1]$, then by using Gram-schmidt, find the corresponding orthogonal set.

b) Show that the set of functions $\varphi_n(x) = \cos(n \cos^{-1} x)$, $n \geq 1$ is orthogonal on $\mathcal{L}_w^2[-1, 1]$, where $w(x) = (1 - x^2)^{-\frac{1}{2}}$. Obtain $\|\varphi_n\|$.

Question 2. a) Write the following differential equations in Sturm-Liouville form

$$\begin{aligned}\mathcal{L}_1 u &= \frac{\sin^2 x}{\cos x} u'' - \frac{1}{\cos x} u' + (\sin x) u = 0 \\ \mathcal{L}_2 u &= (x^2 + 1) u'' - 2x u' + (\cos x) u = 0\end{aligned}$$

b) Find α and β such that the functions $F(x) = e^{-\beta x}$, $G(x) = \sqrt{1+x} e^{(\alpha-2)x}$ belong to $\mathcal{L}^2(0, \infty)$.

Question 3 Find the eigenvalues and eigenfunctions of the boundary value problem

$$\begin{cases} u'' + \lambda u = 0, & x \in (-2, 2) \\ u(-2) = u(2), & u'(-2) = u'(2) \end{cases}$$

What are the eigenfunctions $(u_n(x))_{n \geq 1}$ that satisfy $\int_{-2}^2 u_n^2(x) dx = 1$.