King Saud University	College of Sciences		Department of Mathematics
First Examination	Math 132	Semester I (1446)	Time:1H30

Question 1: (10 marks)

1. Decide whether the following propositions is a tautology or a contradiction or a contingency? $(p \to \neg q) \to (r \wedge \neg p)$. (3 marks)
2. Without using truth tables, prove that the following conditional statement is a Tautology:
$$[(p \vee q) \wedge \neg p] \to q. \quad \text{(3 marks)}$$
3. Without using truth tables, prove the following logical equivalence:
$$(p \to q) \to r \equiv (\neg r \to p) \wedge (q \to r). \quad \text{(3 marks)}$$
4. Determine the truth value of each of the following statements. (Justify your answer) (1 mark)
   (a) $\forall x \in \mathbb{R}; (x^2 < x^4)$.
   (b) $\exists x \in \mathbb{R}; (x^2 + 1 = 0)$.

Question 2: (10 marks)

1. Use a proof by contradiction to show that $\dfrac{\sqrt{5}-5}{3}$ is irrational. (Hint use the fact that $\sqrt{5}$ is irrational). (2 marks)
2. Let x, y and z be three real numbers. Use a proof by contraposition to show that: if $(2x - 4y + 5z = 8)$ then, $(x \leq 5$ or $y \geq 3$ or $z \leq 2)$. (2 marks)
3. Use mathematical induction to prove the following statement:
$$8 + 20 + 32 + \cdots + (12n - 4) = 6n^2 + 2n, \quad \text{for each integer } n, \text{ with } n \geq 1. \quad \text{(3 marks)}$$
4. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_1 = 3 \\ u_2 = 6 \\ u_{n+1} = 2u_n - u_{n-1} + 2; \quad n \geq 2 \end{cases}$
Use mathematical induction to prove the following statement:
$$u_n = n^2 + 2, \quad \text{for each integer } n, \text{ with } n \geq 1. \quad \text{(3 marks)}$$

Question 3: (5 marks)

1. Consider the set $A := \{1, 2, \{1\}, \{2\}, \{1, 2, \emptyset\}, \{1, \{1\}\}, \{2, \{2\}\}, \emptyset, \{\emptyset\}\}$.
Determine whether each of the following four statements is true or false.
(Justify your answer). (2 marks)
   (a) S_1: "$\{1, 2, \emptyset\} \subseteq A$".
   (b) S_2: "$\{1, \{1\}\} \subseteq A$".
   (c) S_3: "$\{1, \{\emptyset\}\} \subseteq A$".
   (d) S_4: "$A \cap \{1, 2, \emptyset, \{\{1\}, \{2\}\}\} = \{1, 2\}$".
2. Consider the following three sets $C := \{1, 2, 3, 4\}$, $D := \{2, 3\}$, and
$E := \{(1, 2), (1, 4), (2, 2), (2, 4), (4, 4), (2, 3)\}$. Find the following sets: (3 marks)
(i) $(C \cap D) \times C$. **(ii)** $E \setminus (C \times D)$. **(iii)** $\{\emptyset\} \times E$.

Q_1 10

1)

3

p	q	r	$\neg q$	$p\to\neg q$	$\neg p$	$r\wedge\neg p$	$(p\to\neg q)\to(r\wedge\neg p)$
T	T	T	F	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	T	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

Contingency.

2)

$$[(p\vee q)\wedge\neg p]\to q \equiv [(p\wedge\neg p)\vee(q\wedge\neg p)]\to q$$
$$\equiv [F\vee(q\wedge\neg p)]\to q$$
$$\equiv (q\wedge\neg p)\to q$$
$$\equiv(\neg q\vee p)\vee q \equiv (\neg q\vee q)\vee p$$
$$\equiv T\vee p\equiv T.$$

3

3)

$$(p\to q)\to r\equiv(\neg p\vee q)\to r$$
$$\equiv(\neg p\to r)\wedge(q\to r)$$
$$\equiv(\neg r\to p)\wedge(q\to r)$$

3

4)

a) $\forall x\in\mathbb{R};\ x^2<x^4$. false.

~~$x=2;\ 2^2=4<2^4=16$~~ 0,5

$x=\frac{1}{2};\ \left(\frac{1}{2}\right)^2=\frac{1}{4}\not<\left(\frac{1}{2}\right)^4=\frac{1}{16}$.

b) $\nexists x\in\mathbb{R};\ x^2+1=0$ false; forall $x\in\mathbb{R}$, $x^2+1\geq 1.$ $\neq 0$.

0,5

Q2) 10

1) By contradiction, we assume that $\frac{\sqrt{5}-5}{3}$ is rational

$\Rightarrow \frac{\sqrt{5}-5}{3} = \frac{a}{b}$; $a, b \in \mathbb{Z}$; $b \neq 0$.

$\Rightarrow \sqrt{5}-5 = \frac{3a}{b} = \cancel{\frac{a-3b}{b}}$

$\Rightarrow \sqrt{5} = \frac{3a+5b}{b} = \frac{x}{y}$ $\quad x = 3a+5b \in \mathbb{Z}$. $\quad y = b \in \mathbb{Z}^*$

2

$\Rightarrow \sqrt{5}$ is rational

contradiction

$\Rightarrow \frac{\sqrt{5}-5}{3}$ is irrational.

2) By contraposition, we ~~assume~~ prove that

if $x > 5$ and $y < 3$ and $z > 2$, then $2x - 4y + 5z \neq 8$

we assume that $x > 5$ and $y < 3$ and $z > 2$

$\Rightarrow 2x > 10$ and $-4y > -12$ and $5z > 10$.

$\Rightarrow 2x - 4y + 5z > \underbrace{10 + 12 + 10}_{8}$

$\Rightarrow 2x - 4y + 5z \neq 8$.

2

3) 3

$P(n): 8+20+32+\dots+(12n-4) = 6n^2+2n.$

B.S: $P(1): 8 = 6+2$ True.

I.S: Let $k \geq 1$. we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$P(k): 8+20+\dots+(12k-4) = 6k^2+2k$.

$P(k+1): 8+20+\dots+(12k+8) = 6(k+1)^2+2(k+1)$.

$$8+20+\dots+(12k-4)+(12k+8) = 6k^2+2k+12k+8.$$
$$= 6(k^2+2k+1)+2k+2$$
$$= 6(k+1)^2+2(k+1)$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1, P(n)$ is true.

4)

3 $P(n): u_n = n^2+2$;

B.S. $P(1): u_1 = 1+2 = 3$ True.

$P(2): u_2 = 4+2 = 6$ True.

I.S. Let $k \geq 2$, we assume that $P(1), P(2), \dots, P(k)$ are true and we prove that $P(k+1)$ is true.

$P(k+1): u_{k+1} = (k+1)^2+2$.

$u_{k+1} = 2u_k - u_{k-1} + 2$.

$P(k)$ true $\Rightarrow u_k = k^2+2$.

$P(k-1)$ true $\Rightarrow u_{k-1} = (k-1)^2+1 = k^2-2k+3$.

$$u_{k+1} = 2k^2+4-k^2+2k-3+2$$
$$= k^2+2k+3 = k^2+2k+1+2$$
$$= (k+1)^2+2$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1$; $P(n)$ is true.

3

$d_3)$ 5

1)

a) True: $1 \in A$, $2 \in A$; $\phi \in A$ 0,5

b) True: $1 \in A$; $\{\{1\}\} \in A$. 0,5

c) True: $\{\phi\} \in A$; $1 \in A$. 0,5

d). false $A \cap = \{1, 2, \phi, \{\{1\}, \{2\}\}\}$ 0,5

$= \{1, 2, \phi\}$.

2)

(i) $C \cap D = \{2, 3\}$.

$(C \cap D) \times C = \{(2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$ 1

(ii)

$C \times D = \{(1,2), (1,3), (2,2), (2,3), (3,2), (3,3), (\cancel{4},2), (4,3)\}$.

$(4,4)$

$E \setminus (C \times D) = \{(1,4), (2,4)\}$ 1

iii) $\{\phi\} \times E = \{(\phi, (1,2)), (\phi, (1,4)), (\phi, (2,2)), (\phi, (2,4)), (\phi, (4,4)), (\phi, (2,3))\}$.

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