King Saud University – Department of Mathematics
Math 473 – Final Exam (1447-1)
Duration: 3 hours – Total: 40 points

1. [12 points] Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be the curve given by

$$\alpha(t) = \big(t,\ \frac{t^2}{\sqrt{2}},\ \frac{t^3}{3}\big), \qquad t \in \mathbb{R}.$$

(a) Show that α is regular.
(b) Compute the speed $\|\alpha'(t)\|$ of the curve.
(c) Find the arc-length function $s(t) = \int_0^t \|\alpha'(u)\|\, du$,
(d) Find the length of α on $[0, 3]$.
(e) Find the Frenet-Serret apparatus of α, that is, $T(t)$, $B(t)$, $N(t)$.
(f) Find the curvature $\kappa(t)$ and the torsion $\tau(t)$ of α.
(g) Is α a general helix? Justify.

2. [14 points] Consider the simple surface M given by

$$X(u, v) = (u,\ v,\ u^2 + v^2 + 1).$$

(a) Compute the first fundamental form I.
(b) Find the area of the region $X(\mathcal{R})$, where $\mathcal{R} = \{(u, v) :\ u^2 + v^2 \leq 1\}$.
(c) Compute the normal $\vec{n}$ to M.
(d) Compute the second fundamental form II.
(e) Compute the shape operator L.
(f) Compute the mean curvature H and the Gauss curvatue K_G.
(g) Determine the planar, elliptic, and hyperbolic points of M.

3. [6 points] Let M be a simple surface and let $I = (g_{ij})$ denote its first fundamental form. Show that the Christoffel symbols Γ_{ij}^k of M are given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_\ell} + \frac{\partial g_{i\ell}}{\partial u_j} \right),$$

where the coefficients g^{ij} belong to the inverse matrix I^{-1}.

4. [4 points] Let M be the simple surface with first fundamental form

$$\mathrm{I} = \sin v^2\, du^2 + dv^2, \qquad u \in (0, 2\pi),\ v \in (0, \pi).$$

Compute the Christoffel symbols Γ_{12}^1, Γ_{12}^2.

5. [4 points] Let S^2 be the unit sphere and consider the curve (circle of latitude)

$$\alpha(t) = (\cos\varphi \cos t,\ \cos\varphi \sin t,\ \sin\varphi), \qquad t \in [0, 2\pi),$$

with fixed $\varphi \in (0, \pi/2)$.

(a) Compute the geodesic curvature k_g of α as a curve on the sphere.
(b) For which φ is α a geodesic?