King Saud University
College of Science
Department of Mathematics
2^{nd} Sem. 1447 H. (25/26)

Math 380
Stochastic Processes
First Midterm Exam
Time: 90 Minutes

Problem 1 [4 marks]:
An insurance company classifies its claims as low if they are under \$12000, and high otherwise. During the year, 80% of its policyholders filed no claims, 16% filed low claims, and 4% filed high claims. If a policyholder filed a claim, what is the probability that it was a high claim?

Problem 2 [3 marks]:
Policyholders of an insurance company file claims at an average rate of 0.4 per year. If the company pays \$7500 for each claim, what is the mean claim amount for a policyholder in a year?

Problem 3: [6 marks]
Let X and Y be two random variables with joint density function:

$$f_{XY}(x,y) = \begin{cases} \alpha xy, & \text{for } 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the value of the constant α so that f_{XY} defines a joint density function.
2. Calculate the joint probability $\mathbb{P}\left(X \leq \frac{1}{2}\ ,\ Y \geq \frac{1}{4}\right)$.
3. Find the marginal densities $f_X(x)$ and $f_Y(y)$.
4. Are X and Y independent?
5. Calculate the conditional expectation $\mathbb{E}[Y|X=x]$.

Problem 4 [6 marks]:
Suppose that $\xi_1, \xi_2, \ldots,$ are independent identically distributed (i.i.d.) random variables with common distribution $\mathbb{P}(\xi_i = 1) = p > 0$ and $\mathbb{P}(\xi_i = -1) = 1 - p = q$, and consider their partial sum defined by:

$$\begin{cases} S_0 := 0 \\ S_n := \displaystyle\sum_{j=1}^{n} \xi_j, \ n \geq 1. \end{cases}$$

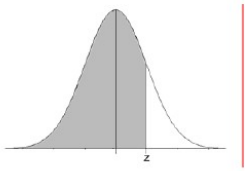
1. Show that the stochastic process $R_n = \left(\frac{q}{p}\right)^{S_n}$ defines a martingale, where $(S_n)_{n\geq 0}$ is as above. (Justify your answer).
2. Consider the stochastic process $Y_n := S_n^2 - \alpha n$, where $(S_n)_{n\geq 0}$ is as above but now with $p = \frac{1}{2}$. Determine for which values of $\alpha \geq 0$ this process Y_n defines a martingale.

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Problem 5: [6 marks]
A company manufactures engines. Specifications require that the length of a certain rod in this engine be between 8.48 cm. and 8.52 cm. The lengths of the rods produced by their supplier have a normal distribution with a mean of $\mu = 8.505$ cm. and a standard deviation of $\sigma = 0.01$ cm.

1. What is the probability that one of these rods meets these specifications?
2. If a worker selects 5 of these rods at random, what is the probability that at least 4 of them meet these specifications?
3. Suppose that, for some technical reason, the lengths' distribution of rods has to be skewed to follow a log-normal distribution with "log-normal" mean 8.50 and standard deviation 0.012. What is the probability that such a rod meets the above required specifications? [Hint: recall that if $V = e^X$, where $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}[V] = e^{\mu + \frac{\sigma^2}{2}}$ and $\mathbf{Var}(V) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$].

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

For negative values use $\Phi(-x) = 1 - \Phi(x)$.

Good Luck

Model answers to MT1 Questions
Math 380
2nd Semester 1447H., 2025/2026G

problem 1:

Notations: C = claim, NC = No claim, LC = low claim
HC = high claim.

Given data: (probabilities)

$P(C) = 1 - 0.8 = 0.2$, $P(NC) = 0.8$, $P(LC) = 0.16$, $P(HC) = 0.04$

(4)

We are asked to calculate the conditional probability

$$P(HC|C) = \frac{P(HC, C)}{P(C)} = \frac{0.04}{0.2} = \boxed{0.2}.$$

problem 2:

The average rate of claims is $\lambda = 0.4$ per year.

The cost for each claim is $C = 7500$. Let X be the random number of claims filed by a policyholder per year.

Then, the total amount claimed per a policyholder in a year is $A = X.C = 7500X$

(3)

Thus, the expected claim amount for a policyholder in a year is

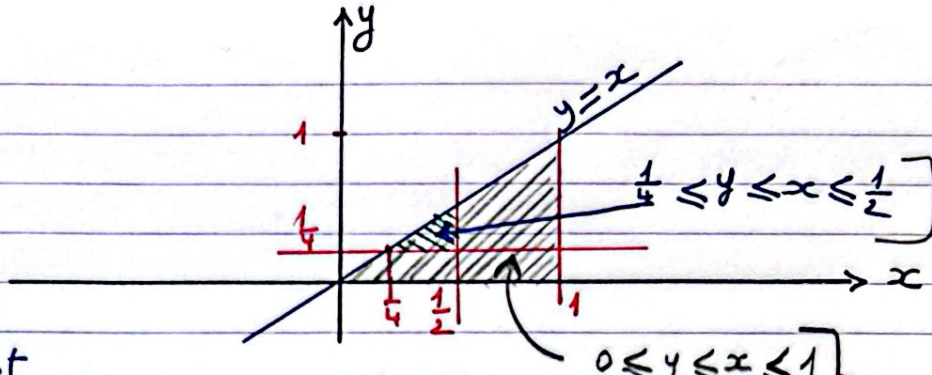
$$E(A) = E(7500X) = 7500\,E(X) = \lambda = 0.4$$
$$= (7500)(0.4)$$
$$\boxed{= 3000\ \$}.$$

problem 3:

1) For f_{XY} to be a density function, we should have

$$\iint_{\mathbb{R}^2} \alpha xy\, dA = 1$$

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so we get

$$\iint_{\mathbb{R}^2} \alpha xy \, dA = \alpha \int_0^1 \int_0^x xy \, dy \, dx$$

$$= \alpha \int_0^1 x \left(\int_0^x y \, dy \right) dx$$

$$= \alpha \int_0^1 x \left[\frac{y^2}{2} \right]_0^x dx = \frac{\alpha}{2} \int_0^1 x^3 dx$$

$$= \frac{\alpha}{2} \left[x^4/4 \right]_0^1 = \frac{\alpha}{2} \left(\frac{1}{4} - 0 \right) = \frac{\alpha}{8} = 1$$

$$\Rightarrow \boxed{\alpha = 8}$$

b

2) $$P\left(X \le \frac{1}{2}, Y \ge \frac{1}{4}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{4}}^{x} 8xy \, dy \, dx$$

$$= 8 \int_{\frac{1}{4}}^{\frac{1}{2}} x \int_{\frac{1}{4}}^{x} y \, dy \, dx$$

$$= 8 \int_{\frac{1}{4}}^{\frac{1}{2}} x \left[\frac{y^2}{2} \right]_{\frac{1}{4}}^{x} dx = 4 \int_{\frac{1}{4}}^{\frac{1}{2}} x \left(x^2 - \frac{1}{16} \right) dx$$

$$= 4 \left[\frac{x^4}{4} - \frac{x^2}{32} \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{1}{16} - \frac{1}{32} - \frac{1}{256} + \frac{1}{128} = \boxed{\frac{9}{256} \approx 0.035}$$

3) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dy = 8x \int_0^x y \, dy$$

$$= 8x \left[\frac{y^2}{2} \right]_0^x = 4x^3 \Rightarrow \boxed{f_X(x) = 4x^3}$$

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$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = 8y \int_y^1 x\,dx$$

$$= 8y\,[x^2/2]_y^1 = 4y(1-y^2)$$

$$\Rightarrow \boxed{f_Y(y) = 4y - 4y^3}$$

4) since $f_{XY}(x,y) = 8xy \neq (4x^3)(4y-4y^3) = f_X(x) f_Y(y)$

we see that X and Y are <u>not independent</u>, (i.e. dependent).

5) $E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X=x}(y)\,dy = \int_{-\infty}^{\infty} y \frac{f_{XY}(x,y)}{f_X(x)}\,dy$

$$= \int_0^x y \frac{8xy}{4x^3}\,dy = \frac{2}{x^2}\int_0^x y^2\,dy$$

$$= \frac{2}{x^2}\left[y^3/3\right]_0^x = \frac{2}{x^2}\frac{x^3}{3} = \frac{2}{3}x$$

$$\Rightarrow \boxed{E[Y|X=x] = \frac{2}{3}x}$$

problem 4 :

1) a) $E[|T_n|] < \infty$?

$$E[|T_n|] = E\left[\left|\left(\tfrac{q}{p}\right)^{S_n}\right|\right] = E\left[\left(\tfrac{q}{p}\right)^{S_n}\right]$$

$$= E\left[\left(\tfrac{q}{p}\right)^{\xi_1}\left(\tfrac{q}{p}\right)^{\xi_2}\cdots\left(\tfrac{q}{p}\right)^{\xi_n}\right] \quad \text{by independence}$$

$$= E\left[\left(\tfrac{q}{p}\right)^{\xi_1}\right].E\left[\left(\tfrac{q}{p}\right)^{\xi_2}\right]\cdots E\left[\left(\tfrac{q}{p}\right)^{\xi_n}\right] \quad (*)$$

But $E\left[\left(\tfrac{q}{p}\right)^{\xi_j}\right] = \left(\tfrac{q}{p}\right)^{(1)}\cdot p + \left(\tfrac{q}{p}\right)^{(-1)}\cdot q \quad , \forall j = 1,2,\ldots,n$ (**)

$$= \frac{q}{p}p + \frac{p}{q}q = q + p = 1 - p + p = 1$$

$(*) \Rightarrow E[|T_n|] = 1.1.\ldots.1 = 1 < \infty.$

b) We have $T_{n+1} = \left(\tfrac{q}{p}\right)^{\xi_{n+1}} T_n$. Let (F_n) be a filtration

$$\Rightarrow E[T_{n+1}|F_n] = E\left[\left(\tfrac{q}{p}\right)^{\xi_{n+1}} T_n \,|\, F_n\right]$$

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$\Rightarrow E[T_{n+1}|F_n] = T_n E[(\frac{q}{p})^{\xi_{n+1}}|F_n]$, T_n is F_n-adapted

$= T_n E[(\frac{q}{p})^{\xi_{n+1}}]$, $(\frac{q}{p})^{\xi_{n+1}}$ is independent of F_n.

$= 1$ by $(\star\star)$

$= T_n$

$\xrightarrow{(a)+(b)}$ $\{T_n\}$ is a martingale.

Remark: one may consider the cases $p = q$ ($\Rightarrow (\frac{q}{p})^{\xi_n} = 1^{\xi_n} = 1$) and $p > q \Rightarrow q > p$ separately, but all lead to the same conclusion that T_n is martingale.

2) For Y_n to be a martingale, it is necessary that

$(\star\star\star)$ $E[Y_{n+1}|F_n] = Y_n$, for a filtration F_n.

But, $Y_{n+1} = X_{n+1}^2 - \alpha(n+1) = (X_n + \xi_{n+1})^2 - \alpha n - \alpha$

$= (X_n^2 - \alpha n) + \xi_{n+1}^2 + 2X_n\xi_{n+1} - \alpha$

$= Y_n + \xi_{n+1}^2 + 2X_n\xi_{n+1} - \alpha$, $E[\alpha|F_n] = \alpha$

$\Rightarrow E[Y_{n+1}|F_n] = E[Y_n|F_n] + E[\xi_{n+1}^2|F_n] + 2E[X_n\xi_{n+1}|F_n] - \alpha$

(Y_n: adapted, ξ_{n+1}^2: independence, $X_n\xi_{n+1}$: adapted + independence)

$= Y_n + E[\xi_{n+1}^2] + 2X_n E[\xi_{n+1}] - \alpha$

$= 1$, $= 0$

$= Y_n + 1 - \alpha = Y_n$ by $(\star\star\star)$

$\Rightarrow \boxed{\alpha = 1}$

Here $E[\xi_j] = (1)(\frac{1}{2}) + (-1)(\frac{1}{2}) = 0$ and

$E[\xi_j^2] = (1)^2(\frac{1}{2}) + (-1)^2(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2} = 1$, $j = 1, 2, \cdots$

problem 5:

1) Let X be the random variable representing the length of a rod. Then, for a product to meet the specifications

we should have $8.48 \leq X \leq 8.52$.

Thus $P(8.48 \leq X \leq 8.52) = P\left(\frac{8.48-8.505}{0.01} \leq Z \leq \frac{8.52-8.505}{0.01}\right)$

$= P(-2.5 \leq Z \leq 1.5)$

$= \Phi(1.5) - \Phi(-2.5)$

$= \Phi(1.5) - 1 + \Phi(2.5)$, $\Phi(-x) = 1 - \Phi(x)$

$\approx 0.9332 - 1 + 0.9938 \approx \boxed{0.927}$.

2) Here we have the Binomial law $P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$ with success probability $p = 0.927$, $n = 5$, Y = number of rods.

So $P(Y \geq 4) = P(Y=4) + P(Y=5)$

$= \binom{5}{4}(0.927)^4(0.073)^{5-4} + \binom{5}{5}(0.927)^5(0.073)^{5-5}$

$= (5)(0.927)^4(0.073) + (1)(0.927)^5(1)$.

$\approx 0.26953 + 0.68453 \approx \boxed{0.9540}$

b

3) Let $V = e^X$, then we find μ and σ (related to X): $\frac{\text{Var}[V]}{E[V]^2} = \frac{e^{2\mu+\sigma^2}(e^{\sigma^2}-1)}{(e^{\mu+\frac{\sigma^2}{2}})^2} = e^{\sigma^2} - 1$

$\Rightarrow e^{\sigma^2} = \frac{(0.012)^2}{(8.5)^2} + 1 \Rightarrow \sigma^2 \approx 0.000000199307$

$\Rightarrow \boxed{\sigma \approx 0.00141176}$

$\mu = \log E[V] - \frac{\sigma^2}{2} \approx 2.140066 - 0.0000009965$

$\Rightarrow \boxed{\mu \approx 2.14006500}$

Therefore

$P(8.48 \leq V \leq 8.52) = P(\ln(8.48) \leq \log V \leq \ln 8.52)$

$= P\left(\frac{\ln(8.48) - 2.140065}{0.00141176} \leq Z \leq \frac{\ln(8.52) - 2.140065}{0.00141176}\right)$

$= P(-1.668 \leq Z \leq 1.665)$

$\approx \Phi(1.665) + \Phi(1.668) - 1$

$\approx 0.9515 + 0.9525 - 1 \approx \boxed{0.9044}$

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