Math 373 -- Final Exam

King Saud University
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<u>Question 1</u>: [12 Marks] Let $\mathcal{L} = \{\mathbb{R}, \emptyset\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$.

a) Prove that $\mathcal{L}$ is a topology on $\mathbb{R}$. [4]

b) Compare between $\mathcal{L}$ and the usual topology $\mathcal{U}$ on $\mathbb{R}$. [1]

c) Find all closed subsets in $(\mathbb{R}, \mathcal{L})$. [1.5]

d) Using (c), find the closure of the subsets: $[0,2), (1,\infty), (-\infty,0)$. [1.5]

e) If $A = (0,1)$, find the subspace topology $\mathcal{L}_A$.[1]

f) Is $(\mathbb{R}, \mathcal{L})$ Hausdorff? Why? [1.5]

g) Is $(\mathbb{R}, \mathcal{L})$ metrizable? Why? [1.5]

<u>Question 2:</u> [6 Marks] Consider $\mathbb{R}$ with the usual topology $\mathcal{U}$, and let $A = (0,\infty)$ and $B = [1,2)$.

a) Show that A is an open set. [1]

b) Show that B is not an open set [1]

c) Find $Int(A), Int(B)$, $Ext(A)$, $Ext(B)$, $Bd(A), Bd(B), A'$, and B'. [4]

Question 3: [2.5 Marks] Let $X = \mathbb{R}$ with the usual topology $\mathcal{U}$, and $Y = \mathbb{R}$ with the half-closed interval topology $\mathcal{H}$.

a) Give a basis for the product topology on $X \times Y$ [1.5]

b) Find $Int([0,1) \times (1,2))$ [1]

Question 4: [3.5 Marks]

a) Let $X = \{a, b, c, d, e\}$ and let $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{d, e\}\}$. Find the topology $\mathcal{T}(\mathcal{S})$ on X generated by $\mathcal{S}$. [2]

b) Show that $\mathcal{B} = \{(a - r, a + r): a \in \mathbb{R}, r > 0\}$ is a base for the usual topology $\mathcal{U}$ on $\mathbb{R}$.[1.5]

Question 5: [3 Marks] Let $f:(\mathbb{R},\mathcal{U}) \longrightarrow (\mathbb{R},\mathcal{U})$ be the function given by

$$f(x) = \begin{cases} x+1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Is f continuous? Is f open? Justify your answer.

Question 6: [5 Marks] Let $X = \mathbb{R}^2$. Define $d: X \times X \longrightarrow \mathbb{R}$ by $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$.

a. Show that d is a metric for X. [3]

b. Find and sketch $B_1^d((0,0))$. [2]

Question 7: [8 Marks]

a. Show that a closed subset of a compact topological space is compact. [2]

b. Let $(X, \mathcal{T})$ be a compact topological space and $f: A \to \mathbb{R}$ a continuous function. Show that f attains its maximum and minimum [2]

c. Show that $(0,1]$ as a subset of $(\mathbb{R}, \mathcal{U})$ is not sequentially compact. [2]

d. Show that every sequentially compact space is limit point compact. [2]