King Saud University – Department of Mathematics
Math 373 – Second Midterm Exam (1447-1)
Duration: 1 h 30 mn (Total: 25 points)

Instructions: Answer all questions. Provide clear reasoning and proofs for each result.

1. (3 points)
   (a) Let $X = \{a, b, c, d, e\}$ with the topology $\mathcal{T} = \{\phi, X, \{a, b\}, \{c, d, e\}\}$.
   Describe the subspace topology for the subset $A = \{b, c, e\}$. (1.5 points)
   (b) Let $X = \mathbb{R}$ with the lower-limit topology $\mathcal{H}$.
   Describe the subspace topology for the subset $A = \mathbb{Z}$. (1.5 points)

2. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x < 0, \\ x, & \text{if } x \geq 0 \end{cases}$$

   1. Determine whether f is $\mathcal{U}$–$\mathcal{U}$ continuous. (1 point)
   2. Determine whether f is $\mathcal{U}$–$\mathcal{U}$ open. (1 point)
   3. Determine whether f is $\mathcal{H}$–$\mathcal{U}$ continuous. (1 point)

3. (6 points)
   (a) Give an example of continuous function which is not open. (2 points)
   (b) Give an example of an open and closed function which is not continuous. (2 points)
   (c) In $(\mathbb{R}, \mathcal{U})$, show that $[0, 1) \cong (0, 1]$ by constructing an explicit homeomorphism between the subspaces $[0, 1)$ and $(0, 1]$. (2 points)

4. (6 points) If $A = (0, 1] \subset (\mathbb{R}, \mathcal{H})$ and $B = [0, 1) \subset (\mathbb{R}, \mathcal{U})$, then find

$$\text{Int}(A \times B), \quad \overline{A \times B}, \quad \text{and } \text{Bd}(A \times B).$$

5. (5 points) Prove that being Hausdorff (i.e. T_2) is a topological property.

6. (2 points) Let $X = \{a, b, c\}$ with the topology $\mathcal{T} = \{\phi, X, \{b\}, \{a, c\}\}$.
   Determine whether $(X, \mathcal{T})$ is Hausdorff.