

QUIZ 3 Solutions

MATH343 - Group Theory

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Topic: Homomorphisms, Normal Subgroups, and Finite Abelian Groups

Q1: Let $\varphi : G \rightarrow G'$ be a homomorphism of groups and $N \triangleleft G'$. Is $\varphi^{-1}(N) \leq G$? Is $\varphi^{-1}(N) \triangleleft G$? Justify.

Solution: Part 1: Is $\varphi^{-1}(N)$ a subgroup of G ?

Yes, $\varphi^{-1}(N) \leq G$.

Justification: Let $H = \varphi^{-1}(N) = \{g \in G \mid \varphi(g) \in N\}$. We verify the subgroup criteria:

1. **Identity:** Let e_G and $e_{G'}$ be the identities of G and G' respectively. Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$. Since N is a subgroup of G' , $e_{G'} \in N$. Thus, $e_G \in H$.
2. **Closure:** Let $x, y \in H$. By definition, $\varphi(x) \in N$ and $\varphi(y) \in N$.

$$\varphi(xy) = \varphi(x)\varphi(y)$$

Since N is a subgroup, it is closed under the operation of G' . Therefore, $\varphi(x)\varphi(y) \in N$, which implies $xy \in H$.

3. **Inverses:** Let $x \in H$. Then $\varphi(x) \in N$.

$$\varphi(x^{-1}) = (\varphi(x))^{-1}$$

Since N is a subgroup, it contains the inverse of all its elements. Thus $(\varphi(x))^{-1} \in N$, which implies $x^{-1} \in H$.

Since all criteria are satisfied, $\varphi^{-1}(N) \leq G$.

Part 2: Is $\varphi^{-1}(N)$ a normal subgroup of G ?

Yes, $\varphi^{-1}(N) \triangleleft G$.

Justification: Let $H = \varphi^{-1}(N)$. To prove normality, we must show that for any $h \in H$ and any $g \in G$, the conjugate $ghg^{-1} \in H$.

Let $h \in H$ and $g \in G$. We test if ghg^{-1} is in H by applying φ :

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(h)(\varphi(g))^{-1}$$

We know that: 1. $\varphi(h) \in N$ (since $h \in H$). 2. $\varphi(g) \in G'$. 3. $N \triangleleft G'$ (Given).

By the definition of a normal subgroup ($N \triangleleft G'$), the conjugate of an element in N by any element in G' remains in N . Therefore:

$$\varphi(g)\varphi(h)(\varphi(g))^{-1} \in N$$

This implies that $ghg^{-1} \in \varphi^{-1}(N)$. Thus, $\varphi^{-1}(N) \triangleleft G$.

Q2: Classify all finite abelian groups of order 100. How many non-isomorphic abelian groups of order 10,000?

Solution: Part A: Classification of Abelian Groups of Order 100

By the *Fundamental Theorem of Finite Abelian Groups*, every finite abelian group is isomorphic to a direct product of cyclic groups of prime-power order.

First, we find the prime factorization of the order:

$$|G| = 100 = 2^2 \cdot 5^2$$

We determine the possible isomorphism classes by finding the partitions of the exponents for each prime factor.

1. For the prime factor 2 (exponent 2): The partitions of 2 are:

- $2 \implies \mathbb{Z}_{2^2} = \mathbb{Z}_4$
- $1 + 1 \implies \mathbb{Z}_2 \times \mathbb{Z}_2$

2. For the prime factor 5 (exponent 2): The partitions of 2 are:

- $2 \implies \mathbb{Z}_{5^2} = \mathbb{Z}_{25}$
- $1 + 1 \implies \mathbb{Z}_5 \times \mathbb{Z}_5$

The total number of non-isomorphic groups is the product of the number of partitions for each prime: $p(2) \times p(2) = 2 \times 2 = 4$.

The 4 non-isomorphic abelian groups of order 100 are:

1. $\mathbb{Z}_4 \times \mathbb{Z}_{25} \cong \mathbb{Z}_{100}$
2. $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20} \times \mathbb{Z}_5$

$$3. \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \cong \mathbb{Z}_{50} \times \mathbb{Z}_2$$

$$4. \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10}$$

Part B: Number of Abelian Groups of Order 10,000

First, factor the order:

$$|G| = 10,000 = 10^4 = (2 \cdot 5)^4 = 2^4 \cdot 5^4$$

To find the number of non-isomorphic groups, we calculate $p(4)$, the number of partitions of the integer 4, for each prime power.

The partitions of 4 are:

- 4
- 3 + 1
- 2 + 2
- 2 + 1 + 1
- 1 + 1 + 1 + 1

Counting these, we find that $p(4) = 5$.

Since the exponents for both prime factors (2 and 5) are 4, the total number of non-isomorphic abelian groups is:

$$p(4) \times p(4) = 5 \times 5 = 25$$

Answer: There are **25** non-isomorphic abelian groups of order 10,000.
