

Quiz 1 SOLUTIONS

Problem (a): Prove or disprove: $H = \left\{ \frac{a}{2^n} : a, n \in \mathbb{Z}, n \geq 0 \right\}$ forms a group under the usual addition.

Proof:

1. Closure: Let $x = \frac{a}{2^m}$ and $y = \frac{b}{2^k}$ be elements in H .

$$\text{Then } x + y = \frac{a}{2^m} + \frac{b}{2^k} = \frac{a \cdot 2^k + b \cdot 2^m}{2^m \cdot 2^k} = \frac{a \cdot 2^k + b \cdot 2^m}{2^{m+k}}$$

Since $a \cdot 2^k + b \cdot 2^m$ is an integer and $m + k \geq 0$, we have $x + y \in H$. Therefore, H is closed under addition.

2. Associativity: Addition of rational numbers is associative, so this property is inherited.

3. Identity Element: The additive identity is 0, which can be written as $\frac{0}{2^0} = 0 \in H$. For any $x = \frac{a}{2^n} \in H$: $x + 0 = 0 + x = x$.

4. Inverse Elements: For any $x = \frac{a}{2^n} \in H$, consider $-x = \frac{-a}{2^n}$. Since $-a \in \mathbb{Z}$ and $n \geq 0$, we have $-x \in H$. Moreover,
$$x + (-x) = \frac{a}{2^n} + \frac{-a}{2^n} = \frac{a - a}{2^n} = \frac{0}{2^n} = 0.$$

Problem (b): Let $S = \{A \in M_n(\mathbb{R}) : A^T = A\}$. Show that S is a subgroup of $M_n(\mathbb{R})$ under matrix addition. Is $S \cap GL(n, \mathbb{R})$ a subgroup of $GL(n, \mathbb{R})$ under matrix multiplication?

Part 1: S is a subgroup of $(M_n(\mathbb{R}), +)$

Proof: We verify the three subgroup criteria:

1. **Non-empty:** The zero matrix O satisfies $O^T = O$, so $O \in S$.
2. **Closure:** Let $A, B \in S$. Then $A^T = A$ and $B^T = B$.
$$(A + B)^T = A^T + B^T = A + B$$

Therefore, $A + B \in S$.
3. **Inverses:** Let $A \in S$. Then $A^T = A$.
$$(-A)^T = -(A^T) = -A$$

Therefore, $-A \in S$.

Part 2: Is $S \cap GL(n, \mathbb{R})$ a subgroup of $(GL(n, \mathbb{R}), \cdot)$?

Answer: NO, $S \cap GL(n, \mathbb{R})$ is NOT a subgroup of $(GL(n, \mathbb{R}), \cdot)$.

Counterexample: Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Both are symmetric and invertible, but $AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Since $(AB)^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \neq AB$, we have $AB \notin S$, violating closure.

Problem (c): Find $U(18)$. Find the orders of all its elements.

$U(18) = \{1, 5, 7, 11, 13, 17\}$ consists of elements in $\{0, 1, 2, \dots, 17\}$ that are relatively prime to 18.

Orders of Elements

The order of $a \in U(18)$ is the smallest positive integer k such that $a^k \equiv 1 \pmod{18}$.

1. **Order of 1:** $1^1 = 1 \equiv 1 \pmod{18} \rightarrow \text{Order} = 1$

2. **Order of 5:**

- $5^1 = 5$

- $5^2 = 25 \equiv 7 \pmod{18}$

- $5^3 = 5 \cdot 7 = 35 \equiv 17 \pmod{18}$

- $5^4 = 5 \cdot 17 = 85 \equiv 13 \pmod{18}$

- $5^5 = 5 \cdot 13 = 65 \equiv 11 \pmod{18}$

- $5^6 = 5 \cdot 11 = 55 \equiv 1 \pmod{18} \rightarrow \text{Order} = 6$

3. Order of 7:

- $7^1 = 7$
- $7^2 = 49 \equiv 13 \pmod{18}$
- $7^3 = 7 \cdot 13 = 91 \equiv 1 \pmod{18} \rightarrow \text{Order} = 3$

4. Order of 11:

- $11^1 = 11$
- $11^2 = 121 \equiv 13 \pmod{18}$
- $11^3 = 11 \cdot 13 = 143 \equiv 17 \pmod{18}$
- $11^4 = 11 \cdot 17 = 187 \equiv 7 \pmod{18}$
- $11^5 = 11 \cdot 7 = 77 \equiv 5 \pmod{18}$
- $11^6 = 11 \cdot 5 = 55 \equiv 1 \pmod{18} \rightarrow \text{Order} = 6$

5. **Order of 13:**

- $13^1 = 13$
- $13^2 = 169 \equiv 7 \pmod{18}$
- $13^3 = 13 \cdot 7 = 91 \equiv 1 \pmod{18} \rightarrow \text{Order} = 3$

6. **Order of 17:**

- $17^1 = 17 \equiv -1 \pmod{18}$
- $17^2 = (-1)^2 = 1 \pmod{18} \rightarrow \text{Order} = 2$