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|----------|-------------------|----------------|------------------|
| Math 343 | November 17, 2025 | Midterm Exam 2 | Time: 90 minutes |
| Name:    | Student ID #      |                |                  |

| Question # | 1 | 2 | 3 | 4 | 5 | Total |
|------------|---|---|---|---|---|-------|
| Score      |   |   |   |   |   |       |

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

**Q1:** Determine whether each statement is true or false and justify your answer. (**Choose 4 only**) [6 pts]

1. The set of odd permutations is closed under taking inverses.
2. If  $m \neq n$ , then  $U(m) \not\cong U(n)$ .
3. If  $N$  is a normal subgroup of  $G$  such that  $N$  and  $G/N$  are both cyclic then  $G$  must be cyclic.
4. If  $m \mid |G|$  then  $G$  must have a subgroup of order  $m$ .
5. If  $G$  and  $G'$  are groups such that  $G \oplus G'$  is abelian, then  $G$  and  $G'$  are both abelian.
6.  $|(2,4,6)| = 30$  in  $\mathbb{Z}_6 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{12}$ .

1. True: Let  $\sigma$  be odd. Since  $\sigma \cdot \sigma^{-1} = e$  even, then  $\sigma^{-1}$  must be odd.
2. False:  $3 \neq 4$  but  $U(3) \cong U(4) \cong \mathbb{Z}_2$ .
3. False:  $\langle (123) \rangle \triangleleft S_3$  and  $|S_3 / \langle (123) \rangle| = 2 \Rightarrow \langle (123) \rangle$  and  $S_3 / \langle (123) \rangle$  are cyclic. But  $S_3$  is not.
4. False:  $6 \mid |A_4|$  and  $A_4$  has no subgroup of order 6.
5. True: Let  $g_1, g_2 \in G$  and  $g'_1, g'_2 \in G'$ . Since  $G \oplus G'$  is abelian
 
$$(g_1, g'_1) \cdot (g_2, g'_2) = (g_2, g'_2) \cdot (g_1, g'_1)$$

$$\Leftrightarrow (g_1 g_2, g'_1 g'_2) = (g_2 g_1, g'_2 g'_1)$$

$$\Leftrightarrow g_1 g_2 = g_2 g_1 \text{ and } g'_1 g'_2 = g'_2 g'_1$$
6. True:  $|(2,4,6)| = \text{l.c.m.}(|2|, |4|, |6|) = \text{l.c.m.}(3, 5, 2) = 30$



[4pts] Q2: Let  $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} \in S_7$ .

(i) Write the cycle decomposition of  $\alpha$ , find its order and determine if its odd or even.

(ii) write  $\alpha^2$  as a product of disjoint cycles.

(iii) What the maximum possible order for an element in  $S_7$ .

$$(i) \alpha = (17)(26)(354) \quad \frac{1}{2}$$

$$|\alpha| = \text{l.c.m.}(2, 2, 3) = 6 \quad \frac{1}{2}$$

$$\alpha = (17)(26)(34)(35) \Rightarrow \alpha \text{ is even} \quad \frac{1}{2}$$

$$(ii) \alpha^2 = (17)^2 (26)^2 (354)^2 = (345) \quad \frac{1}{2}$$

(iii) Checking for two cycles with total length 7 so that the l.c.m of their orders is max. This is a product disjoint 3-cycle and 4-cycle which has order 12.

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[3pts]Q3: (a) Let  $G$  be a group. Define what it means for (i) a map  $f$  to be an automorphism of  $G$ . (ii) a subgroup  $H$  to be normal.

(b) Compute the group  $\text{Aut}(\mathbb{Z}_{100})$ , is it cyclic? Explain why all its subgroups must be normal.

(a).  $f: G \rightarrow G$  is an automorphism if it is one to one, onto, and operation preserving.

• A subgroup  $H$  of  $G$  is normal if  $aH = Ha$   $\forall a \in G$ .

(b).  $\text{Aut}(\mathbb{Z}_{100}) \cong U(100) \cong U(4) \oplus U(25)$  1

• It is not cyclic because  $|U(4)| = \phi(4) = 2$  and 1  
 $|U(25)| = \phi(25) = 5^2 - 5 = 20$  are not relatively prime.

• But it is abelian so all its subgroups are normal.

$\frac{1}{2}$

[4 pts] Q4: Let  $G$  be a finite group and  $A, B \triangleleft G$ .

(i) Prove that  $A \cap B \triangleleft G$  and  $[G: A \cap B] = [G:A][A: A \cap B]$ .

(ii) Show that the map  $\varphi: G/A \cap B \rightarrow G/A \oplus G/B$  given by  $\varphi(g(A \cap B)) = (gA, gB)$  is a well-defined operation preserving one to one map.

(iii) Show that if  $[G:A]$  and  $[G:B]$  are relatively prime then  $\varphi$  is an isomorphism.

i). The subgroup  $A \cap B$  is normal, since if  $g \in G$  and  $h \in A \cap B$ , then  $ghg^{-1} \in A$  (since  $A \triangleleft G$  and  $h \in A$ ) also  $ghg^{-1} \in B$  (since  $B \triangleleft G$  and  $h \in B$ )  $\Rightarrow ghg^{-1} \in A \cap B$ .

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$$\bullet [G: A \cap B] = \frac{|G|}{|A \cap B|} = \frac{|G|}{|A|} \cdot \frac{|A|}{|A \cap B|} = \frac{|G|}{|A|} \cdot \frac{|A|}{|A \cap B|} =$$

3/4

$$= [G:A] \cdot [A: A \cap B]$$

(ii) •  $\varphi$  is well-defined:  $g_1(A \cap B) = g_2(A \cap B) \Rightarrow g_1 \in g_2(A \cap B) \subseteq g_2 A$   
and  $g_1(A \cap B) \subseteq g_2 B$   
 $\Rightarrow g_1 A = g_2 A$  and  $g_1 B = g_2 B$

1/2

$$\Rightarrow (g_1 A, g_1 B) = (g_2 A, g_2 B) \Rightarrow \varphi(g_1(A \cap B)) = \varphi(g_2(A \cap B))$$

•  $\varphi$  is operation preserving:

$$\varphi(g_1(A \cap B) \cdot g_2(A \cap B)) = \varphi(g_1 g_2(A \cap B)) = (g_1 g_2 A, g_1 g_2 B)$$

$$= (g_1 A g_2 A, g_1 B g_2 B)$$

$$= (g_1 A, g_1 B) \cdot (g_2 A, g_2 B)$$

$$= \varphi(g_1(A \cap B)) \cdot \varphi(g_2(A \cap B))$$

3/4

•  $\varphi$  is 1-1: Assume  $(g_1 A, g_1 B) = (g_2 A, g_2 B)$ . Then  $g_1 A = g_2 A$  and  $g_1 B = g_2 B \Rightarrow g_1 \in g_2 A$  and  $g_1 \in g_2 B \Rightarrow g_1 = g_2 a = g_2 b$   
 $a \in A$  and  $b \in B$ . Cancelling  $g_2 \Rightarrow a = b \in A \cap B \Rightarrow g_1 \in g_2(A \cap B)$   
 $\Rightarrow g_1(A \cap B) = g_2(A \cap B)$ .

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$$(iii) \left. \begin{aligned} [G: A \cap B] &= [G:A] \cdot [A: A \cap B] \\ &= [G:B] \cdot [B: A \cap B] \end{aligned} \right\} \Rightarrow [G:B] \mid [G:A] \cdot [A: A \cap B]$$

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$$\Rightarrow [G:B] \mid [A: A \cap B]$$

$$\Rightarrow [G : A \cap B] = [G : A] \cdot [G : B] \cdot k$$

$\frac{1}{2}$

$$\Rightarrow |G_{A \cap B}| \geq |G_A| \cdot |G_B|$$

$$\text{Since } \phi \text{ is 1-1: } |G_{A \cap B}| \leq |G_A| \cdot |G_B|$$

$$\Rightarrow |G_{A \cap B}| = |G_A| \cdot |G_B| \text{ and so } \phi \text{ is a bijection.}$$

$$\Rightarrow \phi \text{ is an isom.}$$

[3pts]Q5: Show that none of the following groups are isomorphic:

$$\mathbb{Z}_{24}, \mathbb{Z}_6 \oplus \mathbb{Z}_4, S_4, D_{12}, S_3 \oplus \mathbb{Z}_4, D_4 \oplus \mathbb{Z}_3$$

One way to do it is by computing the centers. Since they are respectively

$$\mathbb{Z}_{24}, \mathbb{Z}_6 \oplus \mathbb{Z}_4, \{e\}, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

cyclicnot cyclic

and they are pairwise nonisomorphic since  $\mathbb{Z}_{24}$  is cyclic,  $\mathbb{Z}_6 \oplus \mathbb{Z}_4$  is not cyclic, and the others have different orders 1, 2, 4, and 6 resp.