Math 343 – 14462, April 21, 2025	Midterm Exam 2	Time: 90 minutes
Name:	Student ID #	

Question	1	2	3	4	5	Total
Grade						

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

Q1: Determine whether each statement is true or false and justify your answer. (**Choose 4 only**) [8pts]

1. Every subgroup of S_4 is normal.
2. There exists a finite group G of order 42 with subgroups H and K such that $|H| = 12$, $|K| = 14$ and $|H \cap K| = 2$.
3. There is a nonabelian group of order 11,
4. If every proper subgroup of G is abelian then G must be abelian.
5. The external direct product of cyclic groups must be cyclic.
6. Let G be a group of order pq, where p and q are prime numbers, then every subgroup of G is cyclic.

1. False, take $H = \{1, (12)\} \Rightarrow (13)H = \{(13), (123)\} \neq H(13) = \{(13), (132)\}$.

2. False, since $|H| = 12 \nmid 42 = |G|$.

3. False, since a group of order 11 (a prime) must be cyclic, hence abelian.

4. False, take $G = S_3$. Proper subgroups have orders 1, 2, or 3 and all are cyclic so abelian.

5. False, take $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

6. False, take $G = S_3$. $|S_3| = 2.3$ and $S_3 \leq S_3$ not cyclic.

Q2: **(i)** Let $G = \langle a \rangle$, where $|a| = 24$. list all generators for the subgroup of order 8. [3pts]

(ii) Show that a group G that has more than $p - 1$ elements of order p cannot be cyclic.

(i) $G = \{e, a, a^2, \ldots, a^{23}\}$. If $H \leq G$ and $|H| = 8$, then $H = \langle a^{24/8} \rangle = \langle a^3 \rangle$ and the generators are: a^3, a^9, a^{15}, a^{21}.

(ii) We will prove the contrapositive: If G is cyclic, then the number of elements of order p are at most $p-1$.

If $p \nmid |G|$, then G has 0 zero elements of order p and done. If $p \mid |G|$, then G has a unique subgp of order p and hence it contains all elements of order p. So there will be at most $p-1$ of them.

[3pts] **Q3: (i)** Find all elements in $Aut(\mathbb{Z}_{24})$.

(ii) Find a cyclic subgroup of $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$ of order 12 and a non-cyclic subgroup of order 12.

(i) $Aut(\mathbb{Z}_{24}) = U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\}$

Thm

(ii) A cyclic subgroup of $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$ of order 12 is $\langle (10, 10) \rangle$

Since $|10| = 4$ in $\mathbb{Z}_{40}$ and $|10| = 3$ in $\mathbb{Z}_{30}$ $\Rightarrow |(10,10)| = \text{l.c.m}(4,3) = 12$.

A non-cyclic subgroup of order 12 is given by $\langle 20 \rangle \oplus \langle 5 \rangle$

$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$. It is not cyclic because $\text{g.c.d}(2, 6) \neq 1$.

[3pts] **Q4:** Let ϕ be a group isomorphism from a group G, to a group $\bar{G}$, and let K be a normal subgroup of $\bar{G}$. Show that:

(i) $\phi^{-1}(K)$ is a normal subgroup of G.

(ii) $Ker(\phi)$ is a normal subgroup of G.

(i) ① Since $\phi^{-1}: \bar{G} \longrightarrow G$ is an isom. (Thm) and $K \leq \bar{G}$, then $\phi^{-1}(K) \leq \bar{G}$ (Thm)

② $\phi^{-1}(K) \triangleleft G$:

Let $g \in G$, $x \in \phi^{-1}(K)$. Then $\exists!\ \bar{g} \in \bar{G}$ and $y \in K$ s.t. $\phi^{-1}(\bar{g}) = g$ and $\phi^{-1}(y) = x$. Now,

$$g x g^{-1} = \phi^{-1}(\bar{g}) . \phi^{-1}(y) . (\phi^{-1}(\bar{g}))^{-1}$$

$$= \phi^{-1}(\bar{g} y \bar{g}^{-1}), \quad \text{Since } \phi^{-1} \text{ is an isom.}$$

Since $K \triangleleft \bar{G}$, $\bar{g} y \bar{g}^{-1} \in K \Rightarrow \phi^{-1}(\bar{g} y \bar{g}^{-1}) \in \phi^{-1}(K)$.

(ii) ϕ is 1-1, so $Ker(\phi) = \phi^{-1}(\bar{e}) = \{e\}$ which is normal in G.

[3pts] **Q5**: Let G be a group and let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy | x, y \in G\}$

(i) Prove that G' is normal in G.

(ii) Prove that G/G' is Abelian.

(i) $G' \trianglelefteq G$: For any $g \in G$, and $z = x^{-1}y^{-1}xy$, then $gzg^{-1} = gx^{-1}y^{-1}xyg^{-1} =$
$= (gx^{-1}g^{-1})(gy^{-1}g^{-1})(gxg^{-1})(gyg^{-1}) = (gxg^{-1})^{-1}(gyg^{-1})^{-1}(gxg^{-1})(gyg^{-1})$
has the same form as z. This generalizes for products also since:
$gz_1z_2 \cdots z_n g^{-1} = gz_1g^{-1}\, gz_2g^{-1} \cdots \cdots gz_ng^{-1}$.
where $z_i = x_i^{-1}y_i^{-1}xy$. Each gz_ig^{-1} is of the desired form.

(ii) G/G' is abelian: For any $aG', bG' \in G/G'$, we have:
$aG'\, bG' = bG'\, aG' \iff abG' = baG' \iff G' = b^{-1}a^{-1}baG' \iff b^{-1}a^{-1}ba \in G'$
and this is true by def. of G'.