

Name: Solution Key

Midterm Exam 1

Time: 90 minutes

Student ID #: _____

Question	1	2	3	4	Total
Grade					

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

[5 pts] Q1: Determine whether each statement is true or false and justify your answer. (Choose 4 only)

1. If G is a group such that $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$, then G must be abelian.

Solution: TRUE. By the standard property of group inverses (the “shoes and socks” property), we always have $(xy)^{-1} = y^{-1}x^{-1}$ for any elements x, y in any group. We are given that $(xy)^{-1} = x^{-1}y^{-1}$. Equating the two expressions yields:

$$y^{-1}x^{-1} = x^{-1}y^{-1}$$

Taking the inverse of both sides:

$$\begin{aligned} (y^{-1}x^{-1})^{-1} &= (x^{-1}y^{-1})^{-1} \\ (x^{-1})^{-1}(y^{-1})^{-1} &= (y^{-1})^{-1}(x^{-1})^{-1} \\ xy &= yx \end{aligned}$$

Since $xy = yx$ for all $x, y \in G$, the group G is abelian.

2. There is an abelian group G with $|G| = n$ for any positive integer n .

Solution: TRUE. For any positive integer n , the cyclic group of integers modulo n under addition, denoted as $\mathbb{Z}_n$, has exactly n elements. Since all cyclic groups are abelian, $\mathbb{Z}_n$ is an abelian group of order n .

3. Let G be an abelian group. If $a, b \in G$ both have finite order, then ab must have a finite order.

Solution: TRUE. Let the order of a be m (so $a^m = e$) and the order of b be n (so $b^n = e$). Consider the element ab raised to the power of mn . Because G is abelian, we can rearrange the terms:

$$(ab)^{mn} = a^{mn}b^{mn} = (a^m)^n(b^n)^m = e^n e^m = e \cdot e = e$$

Since there exists a positive integer mn such that $(ab)^{mn} = e$, the order of ab must be finite (specifically, it divides mn).

4. If a group G has an element of order 15, then G must have at least 8 elements of order 15.

Solution: TRUE. If G has an element a of order 15, then a generates a cyclic subgroup $\langle a \rangle$ of order 15 inside G . The number of generators of any finite cyclic group of order n is given by Euler's totient function $\phi(n)$. Thus, the number of generators of $\langle a \rangle$ is:

$$\phi(15) = \phi(3 \times 5) = \phi(3)\phi(5) = (3-1)(5-1) = 2 \times 4 = 8$$

Every generator of this subgroup also has an order of 15. Therefore, G contains at least these 8 elements of order 15.

5. If G is a group and $H, K \leq G$, then $H \cup K \leq G$.

Solution: FALSE. The union of two subgroups is only a subgroup if one is contained within the other ($H \subseteq K$ or $K \subseteq H$).

Counterexample: Let $G = \mathbb{Z}$ under addition. Let $H = 2\mathbb{Z}$ (multiples of 2) and $K = 3\mathbb{Z}$ (multiples of 3). Both are subgroups of $\mathbb{Z}$.

$$2 \in H \implies 2 \in H \cup K$$

$$3 \in K \implies 3 \in H \cup K$$

However, $2 + 3 = 5$, and $5 \notin 2\mathbb{Z}$ and $5 \notin 3\mathbb{Z}$, so $5 \notin H \cup K$. Since $H \cup K$ is not closed under addition, it is not a subgroup.

[5 pts] Q2: Let G be a group and $a \in G$.

(a) i) Define the centralizer $C(a)$.

Solution: The centralizer of a in G is the set of all elements in G that commute with a .

$$C(a) = \{x \in G \mid xa = ax\}$$

ii) Show that $C(a) \leq G$.

Solution: We use the Two-Step Subgroup Test:

1. *Identity:* $ea = a = ae$, so $e \in C(a)$. The set is non-empty.
2. *Closure:* Let $x, y \in C(a)$. Then $xa = ax$ and $ya = ay$.
Consider $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$. Thus, $xy \in C(a)$.
3. *Inverses:* Let $x \in C(a)$. Then $xa = ax$.
Multiply on the left and right by x^{-1} :
 $x^{-1}(xa)x^{-1} = x^{-1}(ax)x^{-1} \implies (x^{-1}x)ax^{-1} = x^{-1}a(xx^{-1}) \implies ax^{-1} = x^{-1}a$.
Thus, $x^{-1} \in C(a)$.

Therefore, $C(a) \leq G$.

iii) Show that $C(a^{-1}) = C(a)$.

Solution: Let $x \in C(a)$. By definition, $xa = ax$.

Multiply both sides on the left and right by a^{-1} :

$$\begin{aligned} a^{-1}(xa)a^{-1} &= a^{-1}(ax)a^{-1} \\ (a^{-1}x)(aa^{-1}) &= (a^{-1}a)(xa^{-1}) \\ a^{-1}x(e) &= (e)xa^{-1} \implies a^{-1}x = xa^{-1} \end{aligned}$$

This means x commutes with a^{-1} , so $x \in C(a^{-1})$. Thus, $C(a) \subseteq C(a^{-1})$.

Since the inverse of a^{-1} is a (i.e., $(a^{-1})^{-1} = a$), we can apply the exact same logic symmetrically to show $C(a^{-1}) \subseteq C((a^{-1})^{-1}) = C(a)$. Therefore, $C(a) = C(a^{-1})$.

(b) i) Define the center $Z(G)$.

Solution: The center of a group G is the set of all elements in G that commute with *every* element in G .

$$Z(G) = \{z \in G \mid zx = xz \text{ for all } x \in G\}$$

ii) Show that $Z(G) \leq G$.

Solution:

1. **Identity:** $ex = x = xe$ for all $x \in G$, so $e \in Z(G)$.
2. **Closure:** Let $a, b \in Z(G)$. For any $x \in G$, $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$. Thus $ab \in Z(G)$.
3. **Inverses:** Let $a \in Z(G)$. Then $ax = xa$ for all $x \in G$. Multiplying on the left and right by a^{-1} gives $a^{-1}(ax)a^{-1} = a^{-1}(xa)a^{-1} \implies xa^{-1} = a^{-1}x$. This holds for all x , so $a^{-1} \in Z(G)$.

Therefore, $Z(G) \leq G$.

iii) Show that $Z(G) \subseteq C(a)$ for any $a \in G$.

Solution: Let $z \in Z(G)$. By the definition of the center, z commutes with *every* element in G . Therefore, z must commute with a , meaning $za = az$. By the definition of the centralizer, this implies $z \in C(a)$. Since this holds for any $z \in Z(G)$, it follows that $Z(G) \subseteq C(a)$.

(c) Give an example of a group G and $a \in G$ such that $\{e\} \subsetneq Z(G) \subsetneq C(a) \subsetneq G$ is a sequence of proper subgroups.

Solution: Let $G = D_4$, the dihedral group of symmetries of a square (order 8). Let standard rotations be $R_0, R_{90}, R_{180}, R_{270}$ and reflections be H, V, D, D' .

The center of D_4 is $Z(D_4) = \{R_0, R_{180}\}$. Notice that $\{R_0\} \subsetneq \{R_0, R_{180}\}$.

Let $a = H$ (horizontal reflection). The centralizer $C(H)$ consists of elements that commute with H . These are R_0, R_{180}, H , and V . So, $C(H) = \{R_0, R_{180}, H, V\}$.

Notice that $Z(D_4) = \{R_0, R_{180}\} \subsetneq \{R_0, R_{180}, H, V\} = C(H)$.

Finally, $C(H)$ has 4 elements, while D_4 has 8, so $C(H) \subsetneq D_4$.

This gives the required proper subgroup sequence: $\{R_0\} \subsetneq Z(D_4) \subsetneq C(H) \subsetneq D_4$.

[5 pts] Q3: Determine if the following groups are cyclic and if so find all its generators:

(1) $\mathbb{Z}$

Solution: YES, Cyclic. Under addition, the integers can be generated entirely by 1 and -1 .
Generators: $\langle 1 \rangle$ and $\langle -1 \rangle$.

(2) $\mathbb{Q}^*$

Solution: NO, Not Cyclic. The group of non-zero rationals under multiplication cannot be generated by a single rational number. If $\mathbb{Q}^* = \langle p/q \rangle$, we could only generate powers of those specific prime factors in p and q , leaving infinitely many other prime numbers that are not generated.

(3) $\mathbb{Z}_{18}$

Solution: YES, Cyclic. This is a standard finite cyclic group under addition modulo 18. An element $k \in \mathbb{Z}_{18}$ is a generator if and only if $\gcd(k, 18) = 1$.

Generators: 1, 5, 7, 11, 13, 17.

(4) $U(35)$

Solution: NO, Not Cyclic. $U(35)$ represents the multiplicative group of integers modulo 35 that are relatively prime to 35. $|U(35)| = \phi(35) = 24$. Note that 6 and 34 ($\equiv -1 \pmod{35}$) both have order 2. By the properties of cyclic groups there can only be 0 or $\phi(2) = 1$ elements of order 2.

(5) D_4

Solution: NO, Not Cyclic. Because all cyclic groups are abelian, and D_4 is non-abelian, D_4 cannot be cyclic.

[5 pts] Q4: In $\mathbb{Z}_{36}$ find:

(1) All elements of order 6.

Solution: In a cyclic group $\mathbb{Z}_n$, one such element of order 6 is $\frac{36}{6} = 6$. The others are found by multiplying numbers in $\{1, \dots, 5\}$ relatively prime to 6. Hence, the elements of order 6 are: **6 and 30.**

(2) All generators of $\langle 8 \rangle$.

Solution: First, $|\langle 8 \rangle| = \frac{36}{\gcd(8, 36)} = 9$, so the number of generators is $\phi(9) = 6$ and they are: $8(1) = 8, 8(2) = 16, 8(4) = 32, 8(5) = 4, 8(7) = 20, 8(8) = 28$.

(3) A generator for $\langle 21 \rangle \cap \langle 10 \rangle$.

Solution: First, simplify the two subgroups:

$\langle 21 \rangle = \langle \gcd(21, 36) \rangle = \langle 3 \rangle$. This is the subgroup of all multiples of 3.

$\langle 10 \rangle = \langle \gcd(10, 36) \rangle = \langle 2 \rangle$. This is the subgroup of all multiples of 2.

The intersection of the subgroup of multiples of 3 and the subgroup of multiples of 2 will simply be the subgroup of multiples of their least common multiple.

$\text{lcm}(3, 2) = 6$.

Therefore, $\langle 21 \rangle \cap \langle 10 \rangle = \langle 6 \rangle$.

A generator for this intersection is **6** (Note: 30 is also a valid generator).