

Mid 1 Solutions

Math 343

Q1: Determine whether each statement is true or false and justify your answer. (Choose 4 only)

[8 pts]

1. In any group G , the set of elements of order at most 2 forms a subgroup.
2. Every group of order 4 is cyclic.
3. Let G be a cyclic group of order n , and $k|n$ a positive integer. Then there exists a subgroup of order k .
4. A group that has no proper non-trivial subgroup is abelian.
5. Let G be a group, $a, b \in G$. Then $(ab)^2 = a^2b^2$ if and only if $ab = ba$.
6. There exists a cyclic group with three different elements of order 3.
7. There exists a group $G = \langle a, b \rangle$ which is not abelian.

1. False: The set of elements of order at most 2 in D_3 is $H = \{e, V_1, V_2, V_3\}$ which is not a subgp of D_3 since for example $V_1 \cdot V_2$ is rotation not equal to e .
2. False: Take for example $U(8) = \{1, 3, 5, 7\}$ not cyclic since $3^2 = 5^2 = 7^2 = e$.

3. True: Let $G = \langle a \rangle \Rightarrow |a| = n$. Since $k \mid n$ a positive integer, then $\frac{n}{k}$ is a positive integer and $|a^{\frac{n}{k}}| = \frac{n}{\text{g.c.d}(\frac{n}{k}, n)} = \frac{n}{\frac{n}{k}} = k \Rightarrow \langle a^{\frac{n}{k}} \rangle$ is a subgroup of order k .

4. True: Let G be such a group. If $G = \{e\}$ then done. Assume $a \in G$ $a \neq e$. Then $\langle a \rangle \leq G \xRightarrow{\text{by assumption}} \langle a \rangle = G \Rightarrow G$ is cyclic $\Rightarrow G$ is abelian.

5. True: $ab = ba \Rightarrow (ab)^2 = (ba)^2 = baba = abab = a^2b^2$.
 $(ab)^2 = a^2b^2 \Rightarrow \cancel{ab}a\cancel{b} = \cancel{a}a\cancel{b}b \Rightarrow ba = ab$.
 ↗ cancellation law

6. False: If G is cyclic and $a \in G$, $|a| = 3$, then $\langle a \rangle$ is the unique subgroup of order 3, which has a, a^2 as the only elements of order 3. (recall, $\phi(3) = 2$)

7. True: Take for example $D_4 = \langle R_{90}, V \rangle$ which is not abelian.

[3pts] Q2: Let G be a group. Prove that

(i) The identity e is unique.

(ii) If $ab = e$ for elements a, b in G , then $a^{-1} = b$.

(iii) If $G = \langle a \rangle$, then G is abelian.

(i) Suppose $e' \in G$ is another identity. Then $e = e \cdot e' = e'$ since e' is an identity and e is an identity.

(ii) If $ab = e$, then multiply on the left by a^{-1} to get $a^{-1}(ab) = a^{-1}e$
 $\Rightarrow (a^{-1}a)b = a^{-1}e \Rightarrow e \cdot b = a^{-1}e \Rightarrow b = a^{-1}$.

(iii) Let $x, y \in G$, then $x = a^i, y = a^j$ for some integers i, j .
 $\Rightarrow x \cdot y = a^i \cdot a^j = a^{i+j} = a^{j+i} = a^j \cdot a^i = y \cdot x$.

[3pts] Q3: Let G be an abelian group and let $H = \{g^3 : g \in G\}$.

(i) Show that $H \leq G$.

(ii) Find the subgroup H for $G = \mathbb{Z}_6$, $G = U(8)$, and $G = \mathbb{Z}_{1000}$.

(i) $e^3 = e \in H \Rightarrow H \neq \emptyset$. Now, let $x, y \in H$. Then $x = g_1^3, y = g_2^3$

$$\Rightarrow x \cdot y^{-1} = g_1^3 \cdot (g_2^3)^{-1} = g_1^3 \cdot (g_2^{-1})^3 = (g_1 \cdot g_2^{-1})^3 = z^3$$

since G is abelian.

$$\Rightarrow x \cdot y^{-1} \in H.$$

$$(ii) \text{ In } \mathbb{Z}_6 : H = \{0^3, 1^3, 2^3, 3^3, 4^3, 5^3\} = \{0, 3, 0, 3, 0, 3\} = \{0, 3\}$$

$$\text{In } U(8) : H = \{1^3, 3^3, 5^3, 7^3\} = \{1, 3, 5, 7\}$$

In $\mathbb{Z}_{1000}$: Since $\mathbb{Z}_{1000} = \langle 1 \rangle$ is cyclic and $3 = 1^3 \in H$ has order 1000

$$\Rightarrow H = \mathbb{Z}_{1000}.$$

[3 pts] Q4: Let G be a group, $g \in G$, and $H \leq G$. Define $K = \{ghg^{-1} : h \in H\}$.

(i) Show that $K \leq G$.

(ii) Show that $|x| = |gxg^{-1}|$ for all $x, g \in G$. Deduce that H is cyclic iff K is cyclic.

i). $e \in H \Rightarrow geg^{-1} = e \in K \Rightarrow K \neq \emptyset$.

. If $gh_1g^{-1}, gh_2g^{-1} \in K$, then $(gh_1g^{-1})(gh_2g^{-1}) = gh_1h_2g^{-1} = gh_3g^{-1} \in K$.

. If $ghg^{-1} \in K$, then $(ghg^{-1})^{-1} = (g^{-1})^{-1}h^{-1}g^{-1} = gh^{-1}g^{-1} \in K$ since $h^{-1} \in H$.

ii) Note that $x^n = e \Leftrightarrow (gxg^{-1})^n = gx^n g^{-1} = geg^{-1} = e$

and $(gxg^{-1})^n = e \Rightarrow gx^n g^{-1} = e \Rightarrow x^n = e$

It follows that x and gxg^{-1} have the same order.

To show H is cyclic iff K is cyclic. Note first that $|H| = |K|$.

This is because $gh_1g^{-1} = gh_2g^{-1} \iff h_1 = h_2$. Now

H is cyclic $\iff |H| = |X| \iff |K| = |gXg^{-1}|$, since $|H| = |K|$

and $|X| = |gXg^{-1}|$.

[3pts] Q5: In $\mathbb{Z}_{24}$

- (i) List all elements of order 6.
- (ii) List all generators of $\langle 8 \rangle$.
- (iii) Find a generator for $\langle 21 \rangle \cap \langle 10 \rangle$.

(i) The number of elements of order 6 in $\mathbb{Z}_{24}$ (cyclic) is $\phi(6) = \phi(3) \cdot \phi(2) = 2 \cdot 1 = 2$.

There are $\frac{24}{6} = 4$ and $4^5 = 5 \cdot 4 = 20$.

(ii) $\langle 8 \rangle = \{0, 8, 16\}$ and the generators are $8, 16 = 8^2 = 2 \cdot 8$.

(iii) $\langle 21 \rangle = \langle \gcd(21, 24) \rangle = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$

$\langle 10 \rangle = \langle \gcd(10, 24) \rangle = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$

$\Rightarrow \langle 21 \rangle \cap \langle 10 \rangle = \{0, 6, 12, 18\} = \langle 6 \rangle$.