Math 343 – 14462, February 17, 2025	Midterm Exam 1	Time: 90 minutes
Name:	Student ID #	

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

Q1: Determine whether each statement is true or false and justify your answer. [5pts]

1. The set of all irrational numbers together with 0 under addition forms a group.
2. There exists $\sigma \in S_n$ such that $\sigma^2 = (12)$.
3. Let G be an abelian group and $x, y \in G$ have finite order, then xy has finite order.
4. A group with only a finite number of subgroups is finite.
5. There is an element of order 4 in $\mathbb{R}^*$ the group of nonzero real numbers under multiplication.

1. False : Not closed, $\underbrace{(1-\sqrt{2})}_{\text{irr.}} + \underbrace{\sqrt{2}}_{\text{irr.}} = \underbrace{1}_{\text{rat.}}$

2. False : $\sigma^2 = \sigma . \sigma$ is even $\forall \sigma \in S_n$ but (12) is odd.

3. True : $(xy)^{|x|\cdot|y|} = x^{|x|\cdot|y|} \cdot y^{|x|\cdot|y|} = (x^{|x|})^{|y|} \cdot (y^{|y|})^{|x|} = e^{|y|} \cdot e^{|x|} = e.e = e.$

Since G is abelian

4. True : Note first that an infinite cyclic group $<a>$ has infinitely many distinct subgps, namely, $<a>, <a^2>, <a^3>, \dots$. Assume G is a gp with finite by many subgp $\Rightarrow$ G has only finite number of cyclic subgp and none of them is infinite. But G is their union, So it must be finite.

5. False, If $x \in \mathbb{R}^*$ and $|x| = 4$, then $x^4 = 1 \Rightarrow x^4 - 1 = 0 \Rightarrow (x^2-1)(x^2+1) = 0 \Rightarrow x = \pm 1$
but $|1| = 1$ and $|-1| = 2$.

[6pts]Q2: a) Show that if $\alpha, \beta \in S_n$ commute and $i \in \{1,2,\ldots,n\}$ is fixed by α , i.e. $\alpha(i) = i$ then $\beta(i)$ is also fixed by α.

$$\alpha(\beta(i)) \underset{\text{def. of "}\circ\text{"}}{=} (\alpha\beta)(i) \underset{\alpha\beta=\beta\alpha}{=} (\beta\alpha)(i) \underset{\text{def. of "}\circ\text{"}}{=} \beta(\alpha(i)) \underset{\alpha(i)=i}{=} \beta(i)$$

b) Let $\sigma = (1643)(3547) \in S_8$.

(1) Write σ as a product of disjoint cycles and find its order.

(2) Write σ as a product of transpositions and determine if σ is even or odd.

(3) Compute σ^{-1} and σ^{22}.

(1) $\sigma = (1647)(35) \Rightarrow |\sigma| = \text{l.c.m}(4,2) = 4.$

(2) $\sigma = (17)(14)(16)(35) \Rightarrow \sigma$ is even.

(3) $\sigma^{-1} = (53)(7461) = (35)(1746)$

$\sigma^{22} = \sigma^{20}\sigma^{2} = (\sigma^{4})^{5}\sigma^{2} = (1)\sigma^{2} = (14)(67)$

[3pts]Q3: Prove that $(\mathbb{Z},*)$ is a group where $a * b = a + b - 1$.

Let a, b, and c be any integers.

① Closure: $a * b = a + b - 1 \in \mathbb{Z}$

② Assoc. : $(a*b)*c = (a+b-1)*c = (a+b-1)+c-1 = a+b+c-2$

$a*(b*c) = a*(b+c-1) = a+(b+c-1)-1 = a+b+c-2$

③ Identity: e is 1, since $a*1 = a+1-1 = a$ and $1*a = 1+a-1 = a$

④ Inverse: a^{-1} is $2-a$, since $a*(2-a) = a+(2-a)-1 = 1 = e$

and $(2-a)*a = (2-a)+a-1 = 1 = e$

[3pts]Q4: a) Let G be a group. (i) Define the center $Z(G)$ and show that it is a subgroup. (ii) Give an example of a group having a proper nontrivial center.

a)(i) Done in class and in the book.

(ii) ① $G = D_4$, since $Z(D_4) = \langle R_{180^\circ} \rangle$.

② $G = GL_2(\mathbb{R})$, Since $GL_2(\mathbb{R})$ is not abelian $Z(GL_2(\mathbb{R})) \neq GL_2(\mathbb{R})$

and Since $-I_2 \in Z(GL_2(\mathbb{R}))$, $Z(GL_2(\mathbb{R})) \neq \{I\}$.

b) Let G be a group and H a nonempty subset that contains $x^{-1}y$ whenever $x, y \in H$. Show H is a subgroup.

① $H \neq \phi$ given.

② Let $x \in H$. $x, x \in H \Rightarrow x^{-1} \cdot x \in H \Rightarrow e \in H$

$\Rightarrow x, e \in H \Rightarrow x^{-1} \cdot e \in H \Rightarrow x^{-1} \in H$.

③ Let $x, y \in H$. Using ②, $x^{-1}, y \in H \Rightarrow (x^{-1})^{-1} \cdot y \in H \Rightarrow x \cdot y \in H$.

[3pts]Q5: Let G be a group.

a) Show that $|ab| = |ba|$ for any $a, b \in G$.

Note that $(ab)^m = e \Longleftrightarrow (ab)^m ab = ab \Longleftrightarrow a(ba)^m b = ab \Longleftrightarrow (ba)^m b = b$
$\Longleftrightarrow (ba)^m = e$.

This shows $|ab|$ is finite iff $|ba|$ is finite and that $|ab| = |ba|$.

b) Show that if $xy = x^{-1}y^{-1}$ for all $x, y \in G$ then G must be abelian.

Note for any $x \in G$, $x, e \in G \Rightarrow x \cdot e = x^{-1} \cdot e^{-1} \Rightarrow x = x^{-1}$

Now, for any $x, y \in G$, we have $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$.