

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

Question	1	2	3	4	5	Total
Grade						40

[9pts] **Q1: Determine whether each statement is true or false and justify your answer. (Choose 6 only)**

1. **Every group of order pq (p, q are twin primes i.e. $q = p + 2$) is abelian. TRUE.** Since p, q are primes and $q = p + 2$, then $q - 1 = p + 1$, it follows that $p \nmid (q - 1)$. By Theorem, $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$, which is cyclic and hence abelian.
2. **It is possible for a quotient group G/N to be abelian even when G is nonabelian. TRUE.** For example, let $G = S_3$ (which is nonabelian) and $N = A_3$. The quotient group S_3/A_3 has order $6/3 = 2$, and any group of order 2 is isomorphic to $\mathbb{Z}_2$, which is abelian.
3. **There exist at least two nonabelian groups of order 50 which are not isomorphic. TRUE.** The order is $50 = 2 \times 5^2$. The groups D_{25} and $D_5 \oplus \mathbb{Z}_5$ are not isomorphic since D_{25} has elements of order 25 and the second group does not (max order is 5).
4. **If $G/Z(G)$ is cyclic, then G is abelian. TRUE.** Let $G/Z(G) = \langle gZ(G) \rangle$. Any two elements $x, y \in G$ can be written in the form $x = g^k z_1$ and $y = g^m z_2$, where $z_1, z_2 \in Z(G)$. Their product is $xy = (g^k z_1)(g^m z_2) = g^{k+m} z_1 z_2$. Since elements in the center commute with everything, we can rearrange this to $g^m z_2 g^k z_1 = yx$. Thus G is abelian.
5. **Every subgroup of index 2 in a finite group is normal. TRUE.** If $[G : H] = 2$, there are exactly two left cosets (H and $G \setminus H$) and two right cosets (H and $G \setminus H$). For any $a \in H$, $aH = H = Ha$. For any $a \notin H$, the left coset must be $G \setminus H$ and the right coset must also be $G \setminus H$. Thus $aH = Ha$ for all $a \in G$, meaning $H \triangleleft G$.
6. **$\text{Inn}(D_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. TRUE.** The inner automorphism group $\text{Inn}(G)$ is isomorphic to $G/Z(G)$. In D_4 , the center $Z(D_4) = \{R_0, R_{180}\}$, which has order 2. Therefore, $|\text{Inn}(D_4)| = 8/2 = 4$. The quotient $D_4/Z(D_4)$ contains no element of order 4 because the square of any element in D_4 is either R_0 or R_{180} , both of which are in $Z(D_4)$. The only group of order 4 with no elements of order 4 is the Klein 4-group, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
7. **The centralizer $C(a)$ of any element a in a group G is normal in G . FALSE.** Consider $G = S_3$ and $a = (1\ 2)$. The centralizer is $C((1\ 2)) = \{id, (1\ 2)\}$. To check for normality, we conjugate by $(1\ 3)$: $(1\ 3)(1\ 2)(1\ 3)^{-1} = (2\ 3)$. Since $(2\ 3) \notin C((1\ 2))$, the centralizer is not a normal subgroup.
8. **A_4 has a subgroup of order 6. FALSE.** A_4 contains no subgroup H of order 6 (if it did, it would have index 2 and be normal, an order argument will show elements of order 3 will map to identity in G/H , so they are all in the kernel which is H . But there are eight 3-cycles in A_4 , which is too many to fit in a subgroup of order 6).
9. **A Sylow p -subgroup is normal iff it is unique. TRUE.** By Sylow's Third Theorem, all Sylow p -subgroups are conjugate. If a Sylow p -subgroup P is unique, it is the only conjugate of itself, so

$gPg^{-1} = P$ for all $g \in G$, meaning it is normal. Conversely, if $P \triangleleft G$, then $gPg^{-1} = P$ for all g , so P has only one conjugate (itself), meaning it is unique.

[7pts] Q2:

a. Define what it means for two groups G and G' to be isomorphic. **Solution:** Two groups G and G' are isomorphic if there exists a bijective map (both injective and surjective) $\varphi : G \rightarrow G'$ that preserves the group operation, meaning $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$.

b. Let $\varphi : G \rightarrow G'$ be a group homomorphism. Show $\text{Im}(\varphi) \leq G'$ but not necessarily normal. **Solution:** To show $\text{Im}(\varphi)$ is a subgroup, as it is clearly not empty, we check closure and inverses. Let $x', y' \in \text{Im}(\varphi)$. There exist $x, y \in G$ such that $\varphi(x) = x'$ and $\varphi(y) = y'$. Then $x'y' = \varphi(x)\varphi(y) = \varphi(xy) \in \text{Im}(\varphi)$, so it is closed. The inverse $(x')^{-1} = (\varphi(x))^{-1} = \varphi(x^{-1}) \in \text{Im}(\varphi)$. Thus, $\text{Im}(\varphi) \leq G'$.

Counterexample for normality: Let $G = \langle (1\ 2) \rangle \leq S_3$, and let $\varphi : G \rightarrow S_3$ be the inclusion map ($\varphi(x) = x$). Then $\text{Im}(\varphi) = \{id, (1\ 2)\}$, which is a subgroup of S_3 but not normal in S_3 .

c. Let $\varphi : G \rightarrow G'$ be a surjective group homomorphism. Show that if G is abelian, then G' is abelian. **Solution:** Let $a', b' \in G'$. Since φ is surjective, there exist $a, b \in G$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. Because G is abelian, $ab = ba$. Therefore, $a'b' = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = b'a'$. Since $a'b' = b'a'$ for all elements in G' , the group G' is abelian.

[8pts] Q3:

a. Let $\alpha \in S_9$ be an element of order 12. List all possible cycle decompositions of α and give an example for each one. **Solution:** The order of a permutation is the least common multiple of the lengths of its disjoint cycles. The lengths must partition an integer ≤ 9 . We seek lengths whose sum ≤ 9 and whose lcm is 12. The only integer partitions satisfying this are:

- Cycle lengths 4 and 3: (sum is $7 \leq 9$, $\text{lcm}(4, 3) = 12$).
Example: $(1\ 2\ 3\ 4)(5\ 6\ 7)$
- Cycle lengths 4, 3, and 2: (sum is $9 \leq 9$, $\text{lcm}(4, 3, 2) = 12$).
Example: $(1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$

b. Let G be a group and let $H, K \triangleleft G$. Prove that $HK \triangleleft G$ **Part 1: Prove that HK is a subgroup of G ($HK \leq G$)**

To show that HK is a subgroup, we can use the Subgroup Test (showing it is non-empty, closed under the group operation, and closed under inverses).

1. Identity (Non-empty):

Since H and K are subgroups of G , they both contain the identity element e . Therefore, $e = e \cdot e \in HK$. This shows HK is non-empty.

2. Closure under the group operation:

Let $x, y \in HK$. By definition, there exist elements $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $x = h_1k_1$ and $y = h_2k_2$. Consider their product:

$$xy = (h_1k_1)(h_2k_2)$$

Because K is normal in G ($K \triangleleft G$), its left and right cosets are equal for any element in G . Specifically, $Kh_2 = h_2K$. This implies that $k_1h_2 = h_2k_3$ for some $k_3 \in K$. Substituting this back into our product gives:

$$xy = h_1(k_1h_2)k_2 = h_1(h_2k_3)k_2 = (h_1h_2)(k_3k_2)$$

Since H and K are subgroups, they are closed. Thus, $h_1h_2 \in H$ and $k_3k_2 \in K$. Therefore, $xy \in HK$, proving closure.

3. Closure under inverses:

Let $x \in HK$, so $x = hk$ for some $h \in H$ and $k \in K$. The inverse is:

$$x^{-1} = (hk)^{-1} = k^{-1}h^{-1}$$

Since $h \in H$ and $k \in K$, their inverses $h^{-1} \in H$ and $k^{-1} \in K$. Thus, $k^{-1}h^{-1} \in KH$. Again, because $K \triangleleft G$, we know that subsets commute, meaning $KH = HK$. Therefore, $k^{-1}h^{-1} \in HK$, meaning $x^{-1} \in HK$.

Since HK contains the identity, is closed under the operation, and contains inverses, $HK \leq G$.

Part 2: Prove that HK is normal in G ($HK \triangleleft G$)

To prove normality, we must show that for any $g \in G$ and any $x \in HK$, the conjugate gxg^{-1} is an element of HK .

Let $g \in G$ and $x \in HK$. By definition, $x = hk$ for some $h \in H$ and $k \in K$. Consider the conjugate:

$$gxg^{-1} = g(hk)g^{-1}$$

We can cleverly insert the identity element $e = g^{-1}g$ between h and k :

$$g(hk)g^{-1} = g(h(g^{-1}g)k)g^{-1} = (ghg^{-1})(gkg^{-1})$$

Now we analyze the two parts of this product:

- Because $H \triangleleft G$, the conjugate ghg^{-1} must be an element of H . Let's call it h' .
- Because $K \triangleleft G$, the conjugate gkg^{-1} must be an element of K . Let's call it k' .

Substituting these back gives:

$$gxg^{-1} = h'k'$$

Since $h' \in H$ and $k' \in K$, the product $h'k'$ is clearly in HK . Therefore, $gxg^{-1} \in HK$, which successfully proves that $HK \triangleleft G$.

c. Show that every subgroup of a cyclic group is cyclic. Solution: Let $G = \langle a \rangle$ be a cyclic group and $H \leq G$. If $H = \{e\}$, it is cyclic (generated by e). If $H \neq \{e\}$, let m be the smallest positive integer such that $a^m \in H$. We claim $H = \langle a^m \rangle$. Let a^k be an arbitrary element of H . By the division algorithm, $k = mq + r$ for some integers q, r with $0 \leq r < m$. Notice that $a^r = a^{k-mq} = a^k(a^m)^{-q}$. Since $a^k \in H$ and $a^m \in H$, closure implies $a^r \in H$. Because m is the minimal positive exponent and $0 \leq r < m$, r must be exactly 0. Therefore, $k = mq$, and $a^k = (a^m)^q \in \langle a^m \rangle$. Hence, H is generated entirely by a^m and is cyclic.

[9pts] Q4:

a. Let $G = \mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$. i. Compute $|(9, 8)|$. Solution: The order of an element (a, b) in a direct product is $\text{lcm}(|a|, |b|)$. In $\mathbb{Z}_{12}$, $|9| = \frac{12}{\gcd(9, 12)} = \frac{12}{3} = 4$. In $\mathbb{Z}_{18}$, $|8| = \frac{18}{\gcd(8, 18)} = \frac{18}{2} = 9$. The order is $\text{lcm}(4, 9) = \mathbf{36}$.

ii. How many elements of order 6 does G have? Solution: We need pairs (x, y) such that $\text{lcm}(|x|, |y|) = 6$. The order $|x|$ must divide 12, and $|y|$ must divide 18. The number of elements of order d in a cyclic group is $\phi(d)$. The pairs $(|x|, |y|)$ that yield an lcm of 6 are:

- $(6, 1) \implies \phi(6)\phi(1) = 2 \times 1 = 2$
- $(6, 2) \implies \phi(6)\phi(2) = 2 \times 1 = 2$
- $(6, 3) \implies \phi(6)\phi(3) = 2 \times 2 = 4$
- $(6, 6) \implies \phi(6)\phi(6) = 2 \times 2 = 4$
- $(1, 6) \implies \phi(1)\phi(6) = 1 \times 2 = 2$
- $(2, 6) \implies \phi(2)\phi(6) = 1 \times 2 = 2$
- $(3, 6) \implies \phi(3)\phi(6) = 2 \times 2 = 4$
- $(2, 3) \implies \phi(2)\phi(3) = 1 \times 2 = 2$
- $(3, 2) \implies \phi(3)\phi(2) = 2 \times 1 = 2$

Total elements of order 6 = $2 + 2 + 4 + 4 + 2 + 2 + 4 + 2 + 2 = \mathbf{24}$.

(Alternative check using subgroups: The number of elements whose order divides 6 is $\gcd(6, 12) \times \gcd(6, 18) = 6 \times 6 = 36$. We subtract elements whose orders divide 2 (which is $\gcd(2, 12) \times \gcd(2, 18) = 2 \times 2 = 4$) and those whose orders divide 3 (which is $\gcd(3, 12) \times \gcd(3, 18) = 3 \times 3 = 9$). By Inclusion-Exclusion, we must add back the identity element we subtracted twice: $36 - 4 - 9 + 1 = 24$.)

b. Let G be an abelian group of order 360: i. Classify (up to isomorphism) all abelian groups of order 360. Solution: The prime factorization of 360 is $2^3 \times 3^2 \times 5$. Using the Fundamental Theorem of Finite Abelian Groups, we partition the prime powers:

- 2^3 : $\mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- 3^2 : $\mathbb{Z}_9, \mathbb{Z}_3 \oplus \mathbb{Z}_3$
- 5^1 : $\mathbb{Z}_5$

Taking all combinations yields 6 distinct isomorphism classes:

1. $\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{360}}$
2. $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{3}} \oplus \mathbb{Z}_{\mathbf{120}}$
3. $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{2}} \oplus \mathbb{Z}_{\mathbf{180}}$
4. $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{6}} \oplus \mathbb{Z}_{\mathbf{60}}$
5. $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{2}} \oplus \mathbb{Z}_{\mathbf{2}} \oplus \mathbb{Z}_{\mathbf{90}}$
6. $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{\mathbf{2}} \oplus \mathbb{Z}_{\mathbf{6}} \oplus \mathbb{Z}_{\mathbf{30}}$

ii. Determine which ones have an element of order 12. Solution: A group has an element of order 12 if its maximum element order is divisible by 12.

- 360 is divisible by 12 (Yes)
- 120 is divisible by 12 (Yes)
- 180 is divisible by 12 (Yes)
- 60 is divisible by 12 (Yes)

- 90 is **not** divisible by 12 (4 does not divide 90) (No)
- 30 is **not** divisible by 12 (4 does not divide 30) (No)

Thus, **groups 1, 2, 3, and 4** have an element of order 12.

c. Prove that a group of order $2p$, where p is an odd prime is isomorphic to $\mathbb{Z}_{2p}$ or D_p .

Solution: Let $|G| = 2p$. By Sylow's Third Theorem, $n_p \equiv 1 \pmod{p}$ and $n_p \mid 2$. Since p is an odd prime, the only possibility is $n_p = 1$. Let $P = \langle a \rangle$ be this unique, normal Sylow p -subgroup of order p . By Cauchy's Theorem, G contains an element b of order 2. Let $K = \langle b \rangle$. Since $P \triangleleft G$, conjugating a by b must yield another element in P : $bab^{-1} = a^k$ for some integer k . Since b has order 2 ($b^2 = e$), applying conjugation twice yields: $b^2ab^{-2} = b(a^k)b^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}$. This implies $a = a^{k^2}$, so $k^2 \equiv 1 \pmod{p}$. The only solutions in $\mathbb{Z}_p$ are $k \equiv 1$ or $k \equiv -1 \pmod{p}$.

- If $k \equiv 1$, then $bab^{-1} = a \implies ba = ab$. G is abelian, generated by elements of coprime orders 2 and p . Thus $G \cong \mathbb{Z}_{2p}$.
- If $k \equiv -1$, then $bab^{-1} = a^{-1}$. This defines exactly the relations of the dihedral group, so $G \cong D_p$.

[7pts] Q5

a. Define a Sylow p -subgroup of a finite group G . **Solution:** Let p be a prime. A Sylow p -subgroup of a finite group G is a subgroup of order p^k , where p^k is the highest power of p that divides the order of G .

b. Let $|G| = 200$ Prove that G is not simple. **Solution:** A group is simple if it has no non-trivial proper normal subgroups. The order is $|G| = 200 = 2^3 \times 5^2$. Let's find the number of Sylow 5-subgroups, n_5 . By Sylow's theorems, $n_5 \equiv 1 \pmod{5}$ and $n_5 \mid 8$. The divisors of 8 are 1, 2, 4, 8. The only divisor that is $1 \pmod{5}$ is 1. Therefore, $n_5 = 1$. This means there is a unique Sylow 5-subgroup (of order 25). Since it is unique, it is a normal subgroup. Because its order (25) is strictly between 1 and 200, it is a proper, non-trivial normal subgroup. Thus, G is not simple.

c. Define what it means for two elements to be conjugates. **Solution:** Two elements $x, y \in G$ are conjugates if there exists some element $g \in G$ such that $gxg^{-1} = y$.

d. Find the conjugacy class of the permutation $\alpha = (1\ 3)$ in S_3 . **Solution:** In symmetric groups, two permutations are conjugate if and only if they have the exact same cycle structure. The permutation $\alpha = (1\ 3)$ is a 2-cycle (a transposition).

Therefore, its conjugacy class is simply the set of all 2-cycles in S_3 . We can directly list them:

$$\{(1\ 2), (1\ 3), (2\ 3)\}$$