Math 343 – 14462, May 18, 2025	Final Exam	Time: 3 hours
Name:	Student ID #	

1. Justify answers and show all work for full credit.
2. Answer the questions in the space provided. If you run out of room continue on the back of the page.

Question	1	2	3	4	5	Total
Grade	7.5	8	7.5	9	8	40

[7.5pts]**Q1**: Determine whether each statement is true or false and justify your answer.

1. The union of two subgroups K, H of a group G is a subgroup.
2. If τ is a cycle, then τ^2 is also a cycle.
3. If N is a normal subgroup of G such N and G/N are both abelian then G must be abelian.
4. There are exactly 36 automorphisms on $\mathbb{Z}_{37}$.
5. A_5 has no subgroup of order 16, 17, 18, or 19.
6. If G is a group and $x, y \in G$ are conjugates, then $|x| = |y|$.

1. False. Take $G = S_3$, $K = \{(1), (12)\}$, $H = \{(1), (13)\}$. Then $K \cup H = \{(1), (12), (13)\} \nleq S_3$
Since $(12)(13) = (132) \notin K \cup H$.

2. False. $(1234)^2 = (13)(24)$

3. False. Take $G = S_3$, $N = A_3$. Then $N \cong \mathbb{Z}_3$ and $G/N \cong \mathbb{Z}_2$ both abelian, but G is not.

4. True. B Thm $Aut(\mathbb{Z}_{37}) = U(37)$ and since 37 is prime, $|U(37)| = 36$.

5. False. Since $|A_5| = 60$ and the numbers 16, 17, 18, and 19 don't divide 60.

6. True. Assume $y = gxg^{-1}$ for some $g \in G$. Then $|x| = |y|$.
$x^n = e \iff e = geg^{-1} = gx^ng^{-1} = \underbrace{gxg^{-1} \cdot gxg^{-1} \cdots gxg^{-1}}_{n\text{-times}} = y^n \implies |x| = |y|$.

[8pts]**Q2**: Let G and G' be groups.

(i) Define what it means for a subgroup of G to be normal. Give an example of a group G and two subgroups, one that is normal and one that is not normal.

(ii) Define what it means for a map $\varphi: G \to G'$ to be a homomorphism and show that the $\ker(\varphi)$ is a normal subgroup of G.

(iii) Show that there is no surjective homomorphism from S_4 onto D_4.

(i) A subgp $H \leq G$ is called normal if $aH = Ha$ for all $a \in G$.

Example. Take $G = S_3$, $N = A_3 \trianglelefteq S_3$ (has index 2), and $H = \{(1),(12)\}$. $H \ntrianglelefteq G$ since

$(13)H = \{(13),(123)\} \neq H(13) = \{(13),(132)\}$.

(ii). A map $\varphi: G \to G'$ is called a homomorphism if $\varphi(g_1g_2) = \varphi(g_1)\cdot\varphi(g_2)$ $\forall g_1,g_2 \in G$.

• $\ker\varphi \trianglelefteq G$: For any $a \in G$ and any $h \in \ker\varphi$, observe that $\varphi(aha^{-1}) = \varphi(a)\cdot\varphi(h)\,\varphi(a)^{-1}$ ($\uparrow$ Since φ is a homo.)

$= \varphi(a)\cdot e\cdot\varphi(a)^{-1}$ ($\uparrow$ since $h \in \ker\varphi$) $= \varphi(a)\cdot\varphi(a)^{-1} = e' \Rightarrow aha^{-1} \in \ker\varphi$

$\overset{\text{by Thm}}{\Longrightarrow} \ker\varphi \trianglelefteq G$.

(iii) If $\varphi: S_4 \to D_4$ is a surjective homo., then by 1st Iso. Thm

$S_4/\ker\varphi \cong \varphi(G) = D_4 \Rightarrow \frac{24}{|\ker\varphi|} = 8 \Rightarrow |\ker\varphi| = 3$. But a subgp of S_4 of order 3 is gen. by a 3-cycle which can't be normal since 3-cycles are conjugates and there are 8 3-cycles.

[7.5pts]**Q3**: Let $\boldsymbol{\alpha} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 1 & 5 & 7 & 8 & 3 \end{pmatrix} \in S_8$

i) Write $\boldsymbol{\alpha}$ as a product of disjoint cycles.
ii) Write $\boldsymbol{\alpha}$ as a product of transpositions (2-cycles).
iii) Deduce if $\boldsymbol{\alpha}$ is an even or odd permutation.
iv) Find $\boldsymbol{\alpha}^{-1}$.
v) Find $\boldsymbol{\alpha}(134)\boldsymbol{\alpha}^{-1}$.

i) $\alpha = (1\,2\,4)(3\,6\,7\,8)$

ii) $\alpha = (1\,4)(1\,2)(3\,8)(3\,7)(3\,6)$

iii) From ii) α is an odd permutation.

iv) $\alpha^{-1} = (1\,4\,2)(3\,8\,7\,6)$

v) $\alpha(1\,3\,4)\alpha^{-1} = (1\,2\,6)$

[9pts]**Q4**: **(i)** In $\mathbb{Z}_{15} \oplus \mathbb{Z}_{16}$, what is $|(6,6)|$?

(ii) Let G be a cyclic group of order 2025. How many elements of order 25 does G have?

(iii) List all elements of order 2 in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$.

(iv) How many abelian groups (up to isomorphism) are there of order 60. List all groups.

(v) Prove that a group of order p^2 is abelian, where p is a prime number.

(vi) Let G be a noncyclic group of order 21. How many Sylow 3-subgroups does G have?

(i) $|(6,6)| = \text{l.cm}(|6|, |6|) = \text{l.cm}(5, 8) = 40$

(ii) Since G is cyclic, $\exists!$ subgp of order 25. So the elements of order 25 are exactly the gen.s for that subgp $\Rightarrow$ we have $\phi(25) = 5^2 - 5 = 20$ such elements.

(iii) $(4,0)$, $(4,1)$, $(0,1)$

(iv) $60 = 2^2 \cdot 3 \cdot 5$. From the Fun. Thm of finite ab gp, we have two non-isomorphic abelian gp.s of order 60, namely: $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

(v) If $|G| = p^2$, then G is a p-gp and by Thm it has a non-trivial center. But then by (Lag. Thm) $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$ then done and if $|Z(G)| = p$, then $G/Z(G)$ is cyclic since $|G/Z(G)| = p$.

So G must be abelian by Thm.

(vi) $|G| = 21 = 3 \cdot 7 \Rightarrow n_3 | 7$ and $n_3 \equiv 1\ (3) \Rightarrow n_3 = 1$ or 7 (note that $n_7 | 3$, $n_7 \equiv 1\ (7) \Rightarrow n_7 = 1$, so Syl 7-subgp is normal). Now if $n_3 = 1$ then G would be cyclic, therefore $\boxed{n_3 = 7}$.

[8pts]**Q5**: **(i)** Classify all finite abelian groups of order 8. Decide with justification which one is isomorphic to $U(15)$.

(ii) Let H be the following subgroup of S_6.

$$\{(1),(12)(34),(1234)(56),(13)(24),(1432)(56),(56)(13),(14)(23),(24)(56)\}.$$

a. Find the order of each element of H.
b. Compute $\sigma\tau$ and $\tau\sigma$, where $\sigma = (1234)(56)$ and $\tau = (56)(13)$.
c. Show that if G is a nonabelian group of order 8, then $|Z(G)|$ must be 2. Classify the group $H/Z(H)$.
d. Up to isomorphism there are only two nonabelian groups of order 8, namely D_4 the dihedral group and $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by $ij = k, jk = i, ki = j$, and $i^2 = j^2 = k^2 = -1$. Explain which one must be isomorphic to H?

(i) There are 3 non-isom. abelian gps of order $8 = 2^3$:

$\mathbb{Z}_8$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Now $U(15) = \{1,2,4,7,8,11,13,14\}$ has more than 2 elements of order 4 including 2, 8, 7. So $U(15)$ must be isom. to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

(ii) a. $|(1)| = 1$, $|(12)(34)| = |(13)(24)| = |(56)(13)| = |(14)(23)| = |(24)(56)| = 2$, $|(1234)(56)| = |(1432)(56)| = \text{l.c.m}(4,2) = 4$.

b. $\sigma\tau = (1234)(56)(56)(13) = (14)(23)$, $\tau\sigma = (56)(13)(1234)(56) = (12)(34)$

c. $|G| = 8 = 2^3$ and G non-abelian $\Rightarrow$ $1 < |Z(G)| < 8 \Rightarrow |Z(G)| = 2$ or 4

but if $|Z(G)| = 4$, then $|G/Z(G)| = 2$ so $G/Z(G)$ is cyclic so G is abelian

$\Rightarrow |Z(G)| = 2.$

d. Since H has two elements of order 4 and Q_8 has 6 namely $\pm i, \pm j, \pm k$, then $H \cong D_4$.