Q1 Find the general solution of the equation

$$y'' + y = 0$$

on $[0, \pi]$, then show that any choice of the initial conditions

$$y(x_0) = \alpha, \quad y'(x_0) = \beta, \quad x_0 \in [0, \pi]$$

gives a unique solution.

Q2 Consider the differential equation

$$u'' - 2xu' + 2nu = 0, \quad x \in \mathbb{R}, \; n \in \mathbb{N}_0. \tag{1}$$

a. Show that the differential operator in (1) is not formally self-adjoint.
b. Transform the differential operator to a formally self-adjoint operator.

Q3 Find the eigenvalues and eigenfunctions of

$$\begin{aligned} u'' + \lambda u &= 0, \quad x \in [0, l], \\ u'(0) &= 0, \quad u'(l) = 0. \end{aligned}$$

Q4 Consider the function

$$f(x) = |x|, \quad x \in [-\pi, \pi]$$

and

$$f(x + 2\pi) = f(x), \; x \in \mathbb{R}.$$

(a) Find the Fourier series representation for f.
(b) Show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$$

(c) Show that the Fourier series converges to $f(x)$ at every $x \in \mathbb{R}$.
(d) Does the Fourier series converge uniformly? Explain?

Q5 Solve the heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \; t > 0,$$

with the following boundary and initial conditions

$$\begin{aligned} u(0, t) &= u(\pi, t) = 0, \quad t > 0, \\ u(x, 0) &= 2, \quad 0 < x < \pi. \end{aligned}$$